IWC 2013
2nd International
Workshop
on Confluence

Workshop on
Infinitary Rewriting
2013

Confluence & Infinitary Rewriting

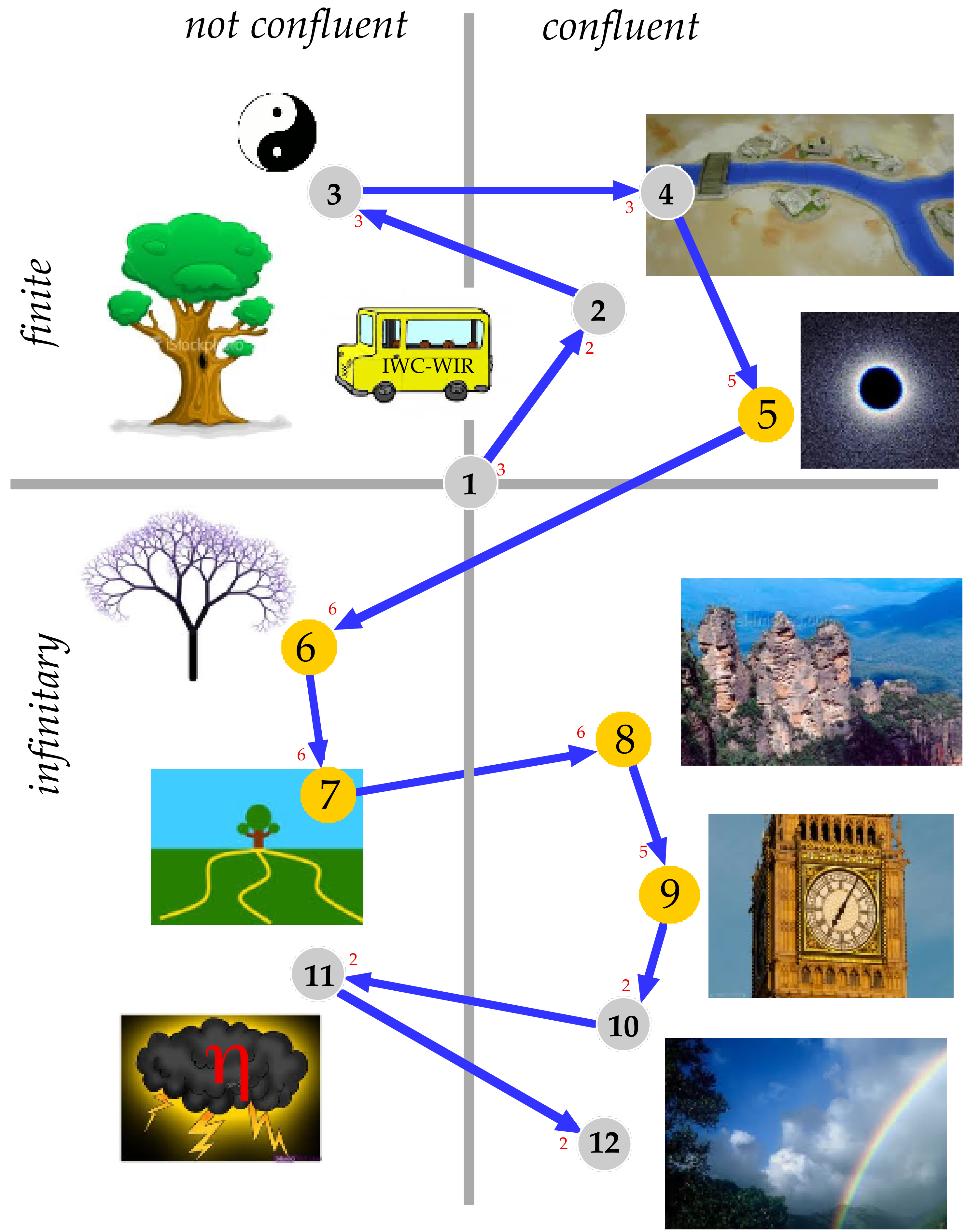
A bus tour with twelve stops

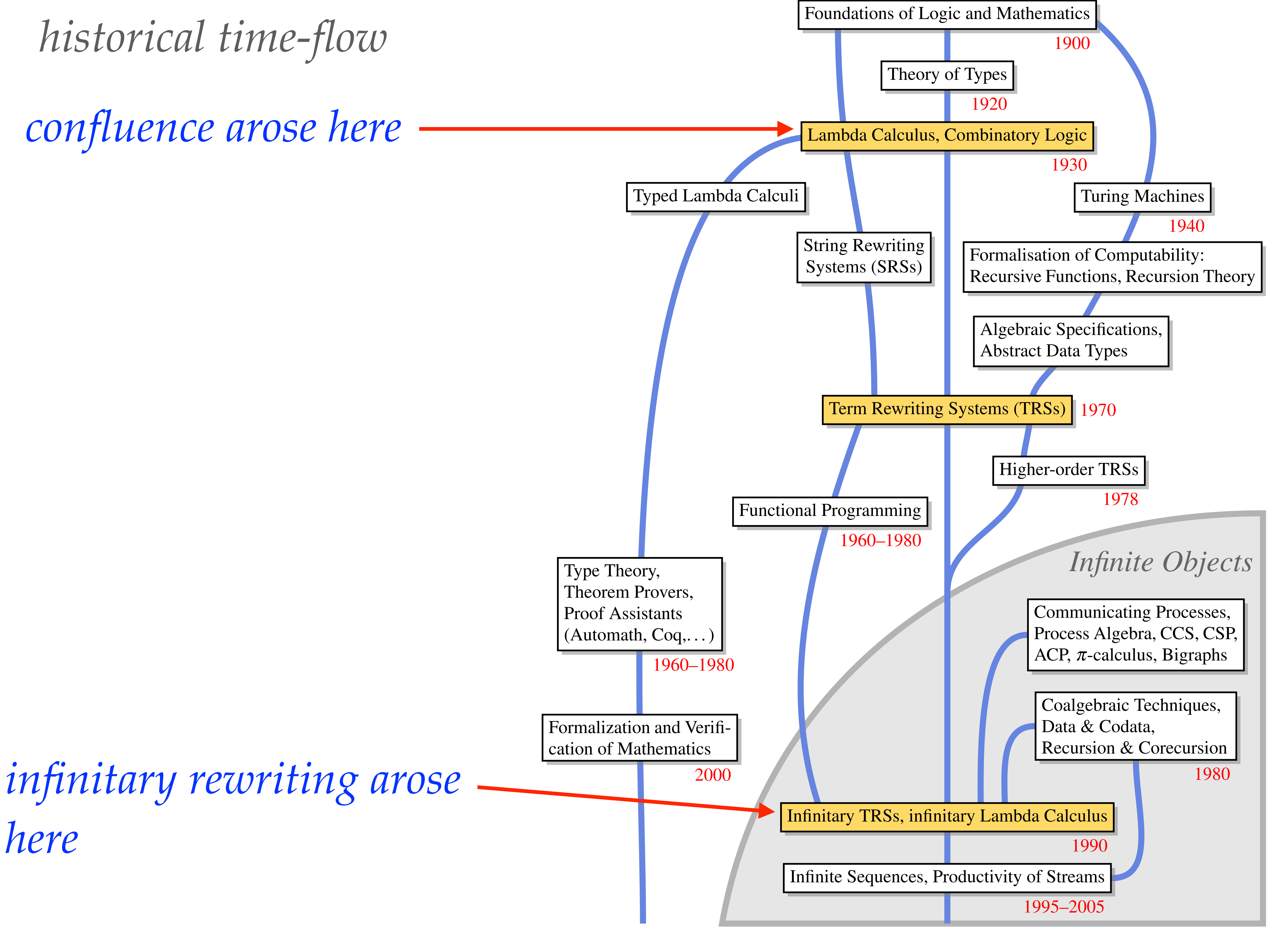
Bus driver: Jan Willem Klop

VU University Amsterdam
CWI Amsterdam

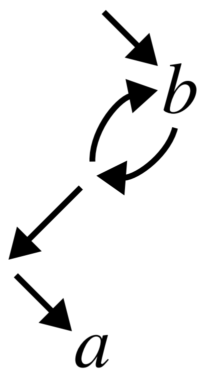
June 28, 2013 Eindhoven

1. *Introduction*
2. *Lambda and CL: basic confluence*
3. *Surjective Pairing: confluence lost*
4. *Confluence of a higher order*
5. *Lambda with black holes: confluence*
6. *Confluence lost in infinity*
7. *The threefold path*
8. *Black holes to the rescue*
9. *The rhythm of lambda terms*
10. *Getting rid of ordinals*
11. *Infinity and eta: total breakdown*
12. *A lambda universe*

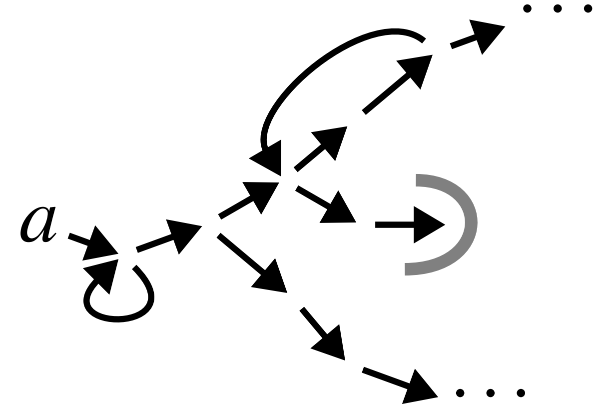




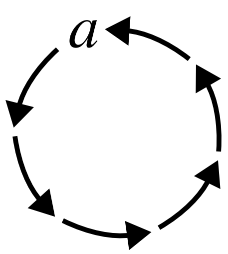
historical time-flow
confluence arose here
infinitary rewriting arose here
Foundations of Logic and Mathematics
1900
Theory of Types
1920
Lambda Calculus, Combinatory Logic
1930
Typed Lambda Calculi
Turing Machines
1940
String Rewriting Systems (SRSs)
Formalisation of Computability: Recursive Functions, Recursion Theory
Algebraic Specifications, Abstract Data Types
Term Rewriting Systems (TRSs)
1970
Higher-order TRSs
1978
Functional Programming
1960–1980
Type Theory, Theorem Provers, Proof Assistants (Automath, Coq,...)
1960–1980
Infinite Objects
Communicating Processes, Process Algebra, CCS, CSP, ACP, π-calculus, Bigraphs
Coalgebraic Techniques, Data & Codata, Recursion & Corecursion
1980
Formalization and Verification of Mathematics
2000
Infinitary TRSs, infinitary Lambda Calculus
1990
Infinite Sequences, Productivity of Streams
1995–2005



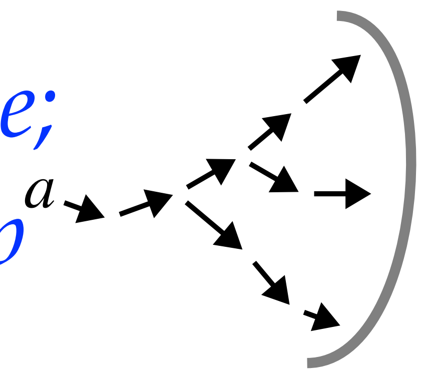
normal form



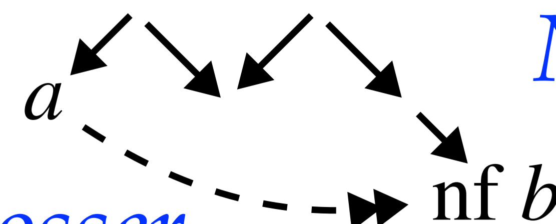
WN, weakly normalizing



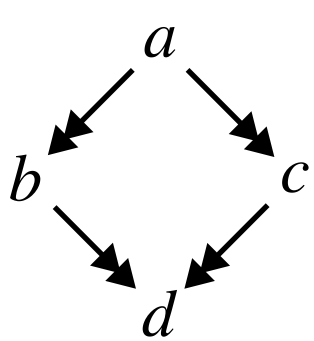
reduction cycle;
loop if one step



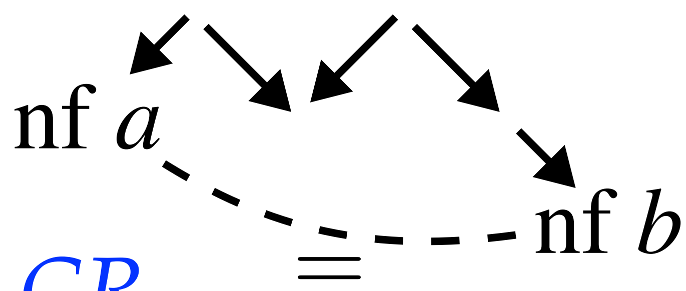
SN, strongly normalizing;terminating; noether



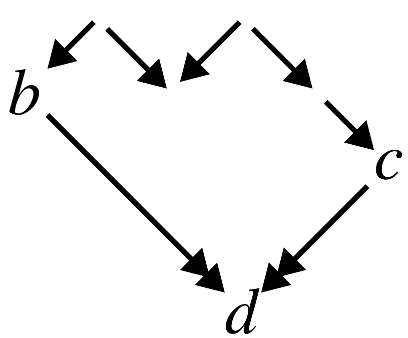
NF, normal form property



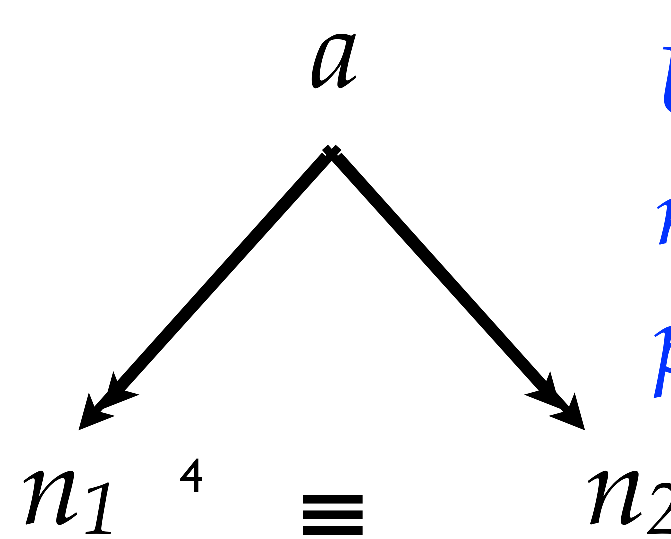
CR, Church-Rosser



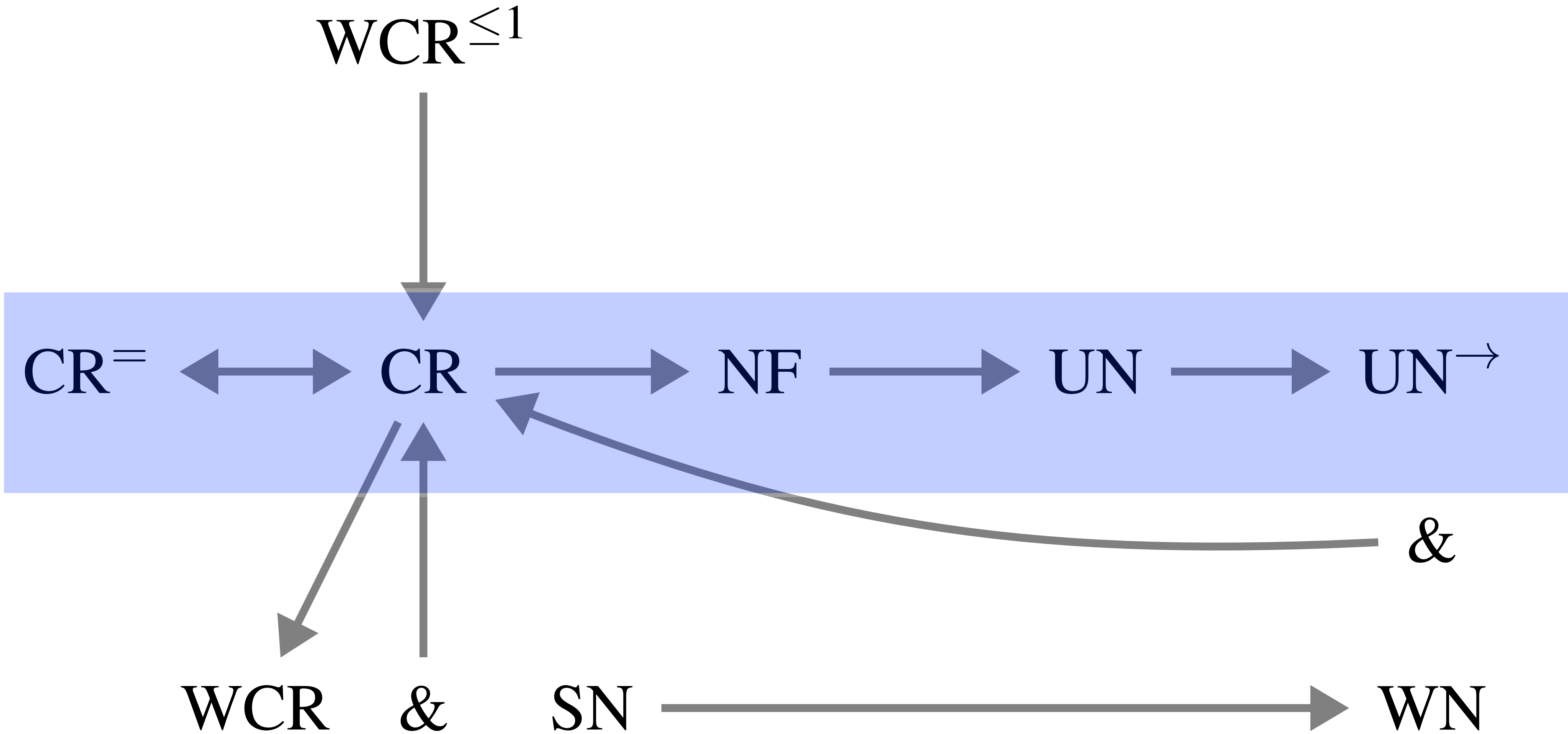
UN⁼, unique normal form property wrt =



equivalent: CR, Church-Rosser



UN→, unique normal form property wrt →

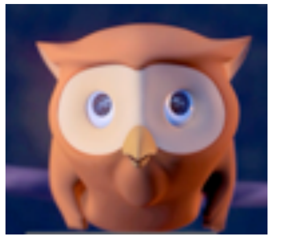


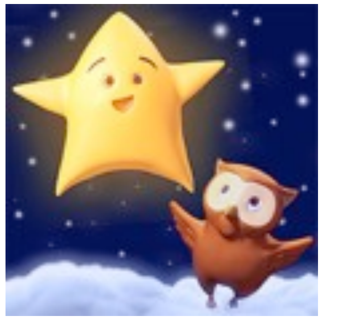
WCR$^{\leq 1}$
CR$^{=}$
CR
NF
UN
UN$^{\rightarrow}$
&
WCR
&
SN
WN

I,K,S, B

$\omega = \lambda x.xx$

$\Omega = \omega\omega$

$\delta = \lambda xy.y(xy) = SI$, Smullyan's Owl 



$\Delta = \delta^{\omega} = \delta(\delta(\delta(\delta \ldots = Y\delta$

$Y_0 = \lambda f. (\lambda x.f(xx)) (\lambda x.f(xx))$ Curry's fpc

$Y_1 = (\lambda ab.\ b(aab)) (\lambda ab.\ b(aab))$ Turing's fpc

$Y_0\ \delta = Y_1$

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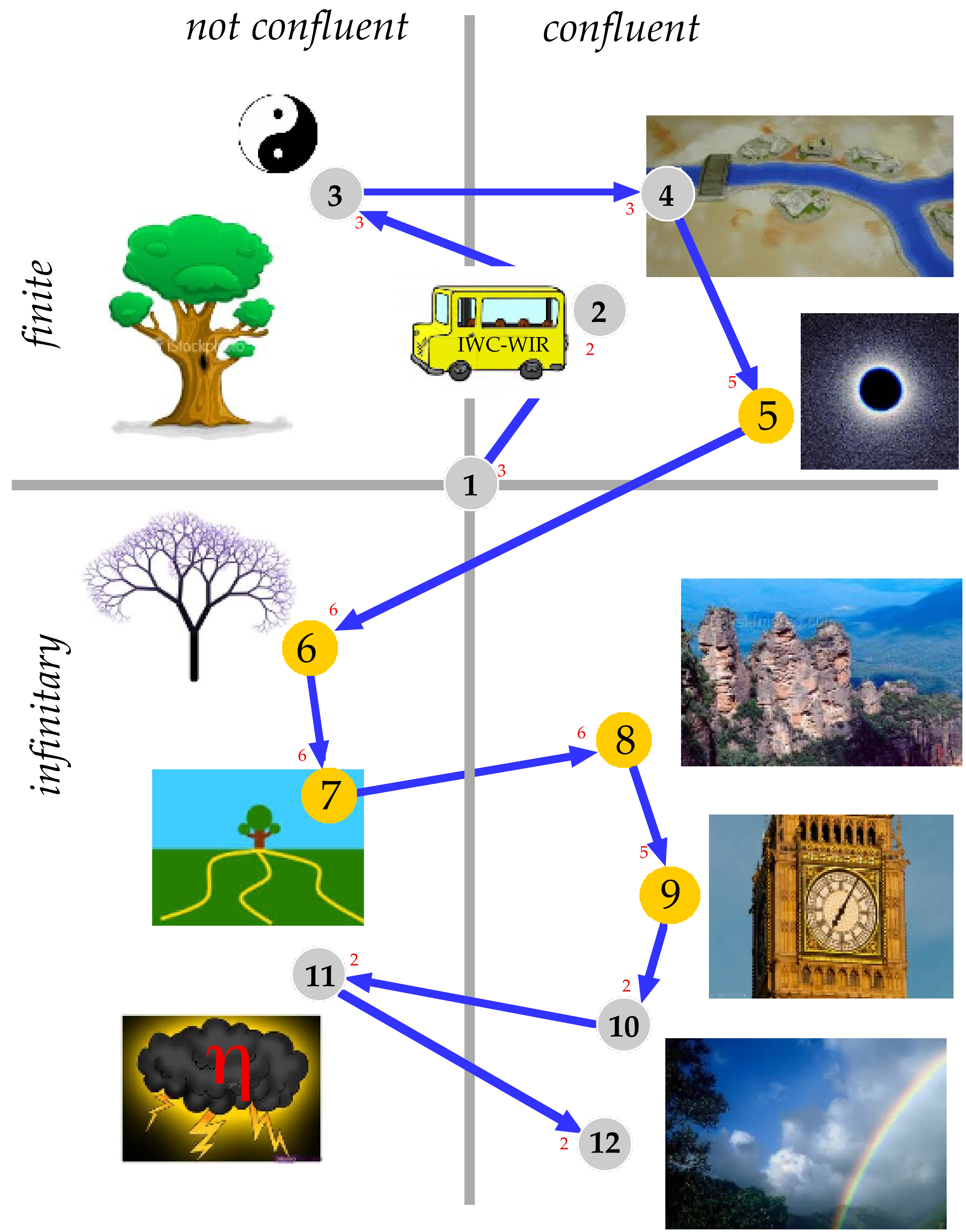


Henk Barendregt, spring 1971

Parallel Reduction à la Tait and Martin–Löf

$$M \multimap\!\!\!\to M$$

$$\frac{M \multimap\!\!\!\to M'}{\lambda x.M \multimap\!\!\!\to \lambda x.M'}$$

$$\frac{M \multimap\!\!\!\to M' \qquad N \multimap\!\!\!\to N'}{MN \multimap\!\!\!\to M'N'}$$

$$\frac{M \multimap\!\!\!\to M' \qquad N \multimap\!\!\!\to N'}{(\lambda x.M)N \multimap\!\!\!\to M'[x:=N']}$$

We use the notation $\multimap\!\!\!\to$ for parallel reduction. In the style of Tait and Martin–Löf, it is defined by the inductive clauses in Table 10. It characterizes complete developments, in the sense that $M \multimap\!\!\!\to N$ if and only if there is a complete development from M to N.

In Aczel [Acz78] the last clause is replaced by:

$$\frac{M \multimap\!\!\!\to \lambda x.M' \qquad N \multimap\!\!\!\to N'}{MN \multimap\!\!\!\to M'[x:=N']}.$$

Now there is a complete *β-super*development form M to N if and only if $M \multimap\!\!\!\to N$ according to Aczel's definition.

EXAMPLE 12.1. In the first definition, due to Tait and Martin-Löf, we do not have $IIII \multimap\!\!\!\to I$ (with $I \equiv \lambda x.x$); in Aczel's definition we do.

Likewise $(\lambda xyz.xyz)\,abc \multimap\!\!\!\to abc$ and even $II(\lambda xyz.xyz)\,abc \multimap\!\!\!\to abc$.

Dear

I would like to mention you a strikingly simple proof of the Church-Rosser theorem for the λ-calculus, due to Martin-Löf in his unpublished paper 'A theory of types',Stockholm 1971 . In this paper an extension of the λ-calculus is considered. However the proof of the Church-Rosser theorem immediately carries over to the λ-calculus itself. The idea of the proof arised from cut-elemination properties of certain formal systems. In fact the Church-Rosser theorem is a kind of cut-elemination theorem, the transitivity of = in the λ-calculus corresponds to the cut.

The trick is to define a relation $\geqslant_1$ between terms in such a way that

1) The transitive closure of $\geqslant_1$ is the (classical) reduction relation ($\geqslant$).
2) If $M_1 \geqslant_1 M_2$, $M_1 \geqslant_1 M_3$, then there exists a term M_4 such that $M_2 \geqslant_1 M_4$ and $M_3 \geqslant_1 M_4$.

From 1) and 2) the analogue of 2) for $\geqslant$ can be derived.
From this the Church-Rosser theorem easily follows.

Now $\geqslant_1$ is defined as follows:

$M \geqslant_1 M$.
If $M \geqslant_1 M'$, then $\lambda xM \geqslant_1 \lambda xM'$.
If $M \geqslant_1 M'$, $N \geqslant_1 N'$ then $MN \geqslant_1 M'N'$.
If $M \geqslant_1 M'$, then $\lambda xM \geqslant \lambda y[x/y]M'$, where $y \notin FV(M')$.
If $M \geqslant_1 M'$, $N \geqslant_1 N'$ then $(\lambda xM)N \geqslant_1 [x/N']M'$ if $FV(N') \cap BV(M') = \emptyset$

($[x/N]M$ stands for the result of substituting N in the free occurrences of x in M ; FV(M) resp. BV(M) is the set of free resp. bound variables of M).

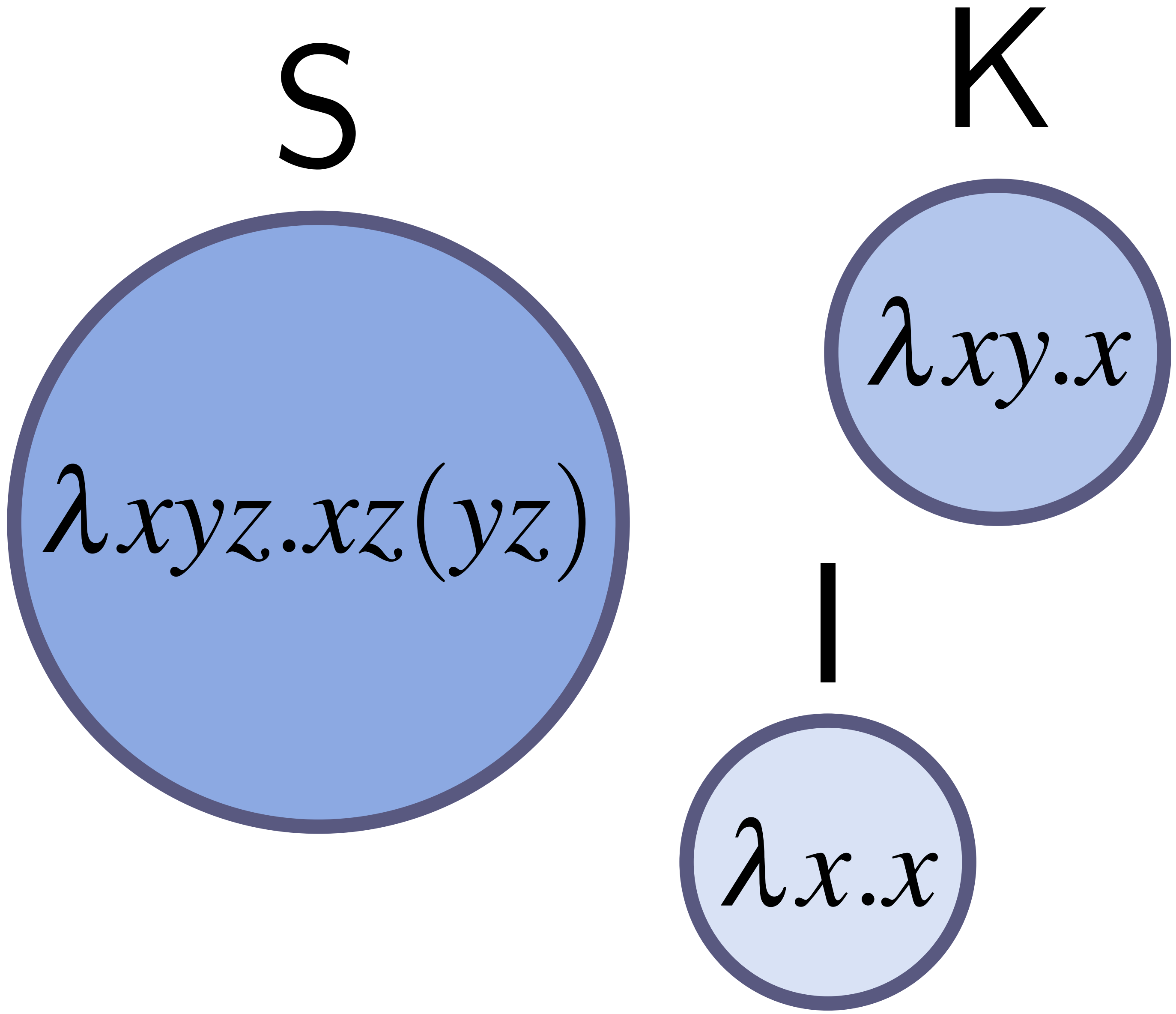
It is clear that $\geqslant_1$ satisfies 1). A simple inductive proof shows that $\geqslant_1$ also satisfies 2).

In the same way the Church-Rosser theorem can be proved when η-reduction is included.

Sincerely yours,

Henk Barendregt.

Mathematisch Instituut
Budapestlaan 6
Utrecht- De Uithof
Holland

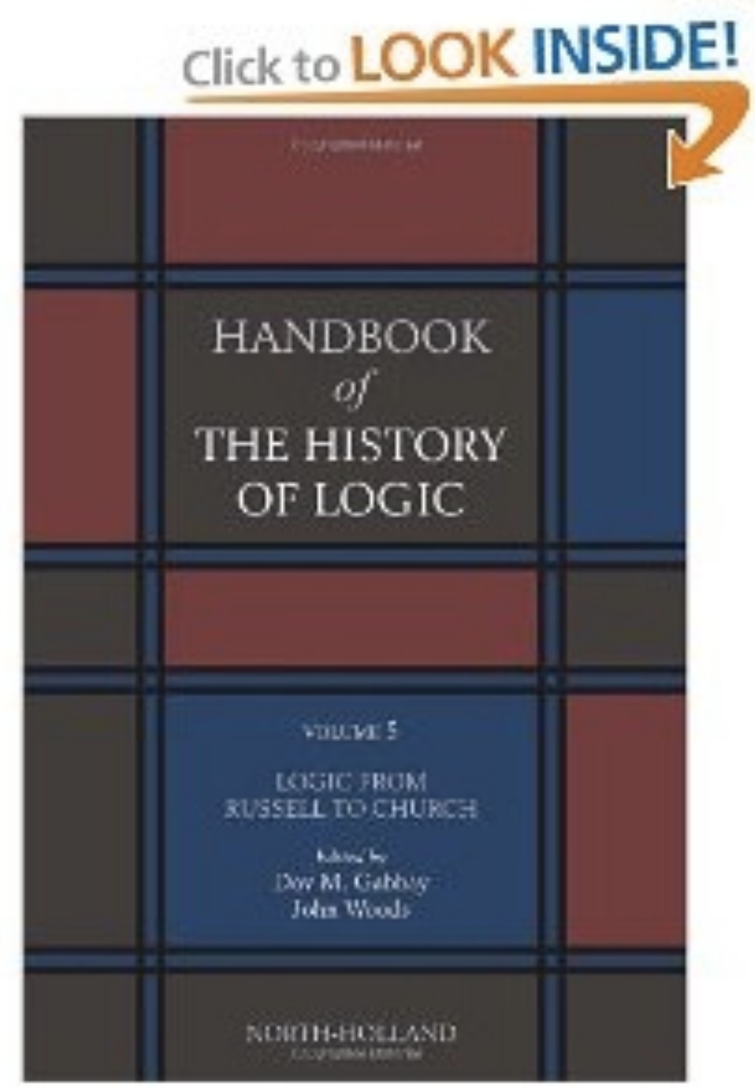


1924. "Über die Bausteine der mathematischen Logik"

Moses Schönfinkel

History of Lambda-calculus and Combinatory Logic

Felice Cardone * J. Roger Hindley †
2006,



Ever since the original proof of the confluence of $\lambda\beta$-reduction in [Church and Rosser, 1936], a general feeling had persisted in the logic community that a shorter proof ought to exist. The work on abstract confluence proofs described in §5.2 did not help, as it was aimed mainly at generality, not at a short proof for $\lambda\beta$ in particular.

For CL, in contrast, the first confluence proof was accepted as reasonably simple; its key idea was to count the simultaneous contraction of a set of non-overlapping redexes as a single unit step, and confluence of sequences of these unit steps was easy to prove, [Rosser, 1935, p.144, Thm. T12].

Then in 1965 William Tait presented a short confluence proof for CL to a seminar on λ organized by Scott and McCarthy at Stanford. Its key was a very neat definition of a unit-step reduction by induction on term-structure. Tait's units were later seen to be essentially the same as Rosser's, but his inductive definition was much more direct. Further, it could be adapted to $\lambda\beta$. (This possibility was noted at the seminar in 1965, see [Tait, 2003, p.755 footnote]). Tait did not publish his method directly, but in the autumn of 1968 he showed his CL proof to Per Martin-Löf, who then adapted it to $\lambda\beta$ in the course of his work on type theory and included the $\lambda\beta$ proof in his manuscript [Martin-Löf, 1971b, pp.8–11, §2.5].

Martin-Löf's $\lambda\beta$-adaptation of Tait's proof was quickly appreciated by other workers in the subject, and appeared in [Barendregt, 1971, Appendix II], [Stenlund, 1972, Ch. 2] and [Hindley *et al.*, 1972, Appendix 1], as well as in a report by Martin-Löf himself, [Martin-Löf, 1972b, §2.4.3].[42]

In λ, each unit step defined by Tait's structural-induction method turned out to be a minimal-first development of a set of redexes (not necessarily disjoint). Curry had introduced such developments in [Curry and Feys, 1958, p.126], but had used them only indirectly; Hindley had used them extensively in his thesis, [Hindley, 1969a, p.547,"MCD"], but only in a very abstract setting. They are now usually called *parallel reductions*, following Masako Takahashi. In [Takahashi, 1989] the Tait-Martin-Löf proof was further refined, and the method of dividing reductions into these unit steps was also applied to simplify proofs of other main theorems on reductions in λ.

Tait's structural-induction method is now the standard way to prove confluence in λ and CL. However, some other proofs give extra insights into reductions that this method does not, see for example the analysis in [Barendregt, 1981, Chs. 3, 11–12].

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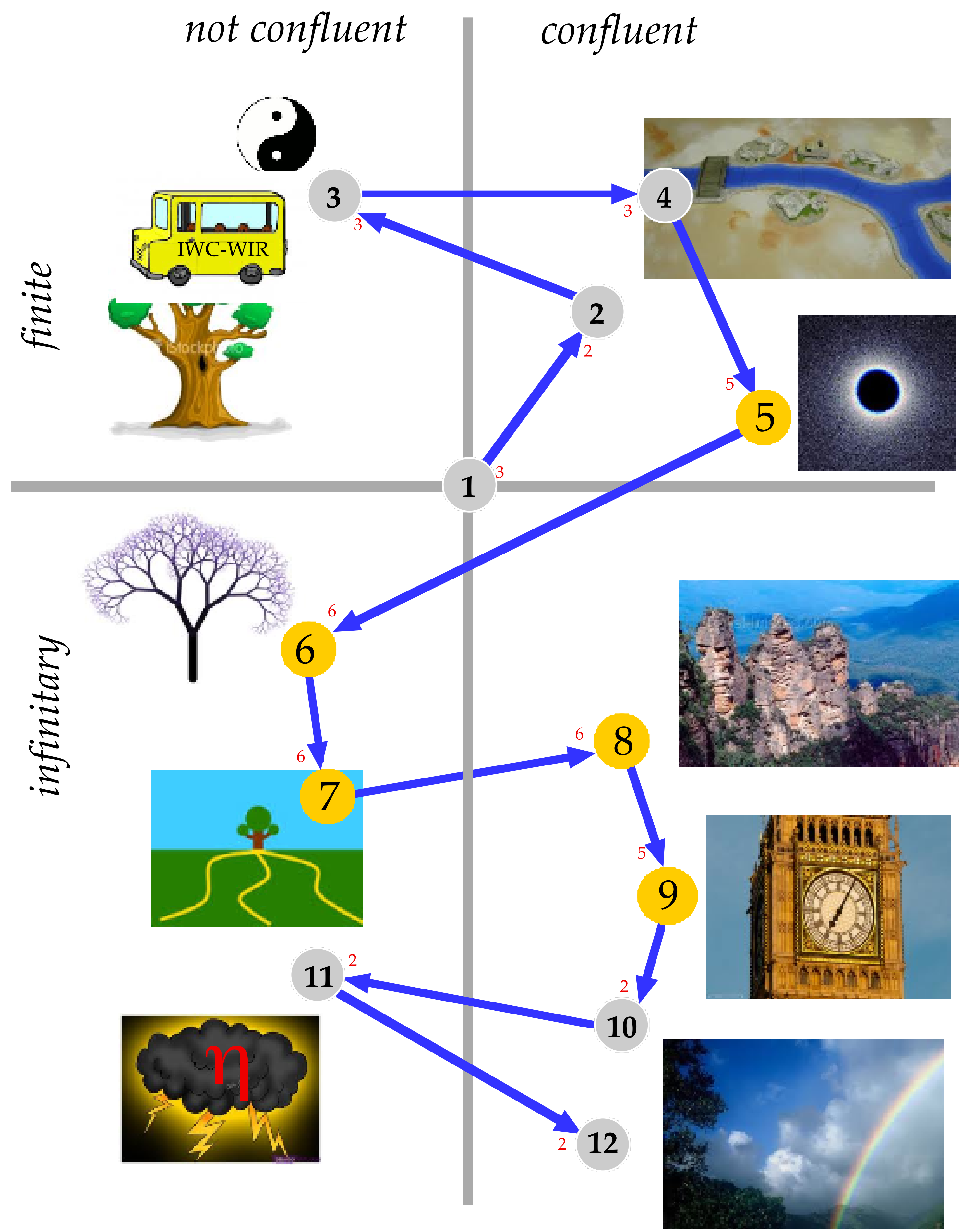


De Vrijer 1989

Extending the lambda calculus with surjective pairing is conservative

Consideration is given to the equational theory $\lambda\pi$ of lambda calculus

extended with constants π, π_0, π_1 and axioms for subjective pairing:

$\pi_0(\pi XY)=X$, $\pi_1(\pi XY)=Y$, $\pi(\pi_0 X)(\pi_1 X)=X$.

nasty overlap

The reduction system that one obtains by reading the equations

are reductions (from left to right) is not Church-Rosser.

Despite this failure, the author obtains a syntactic consistency

proof of $\lambda\pi$ and shows that it is a conservative extension of the pure λ calculus

Klop, de Vrijer 1989: but UN holds

A Question of Balance (The Moody Blues 1970)

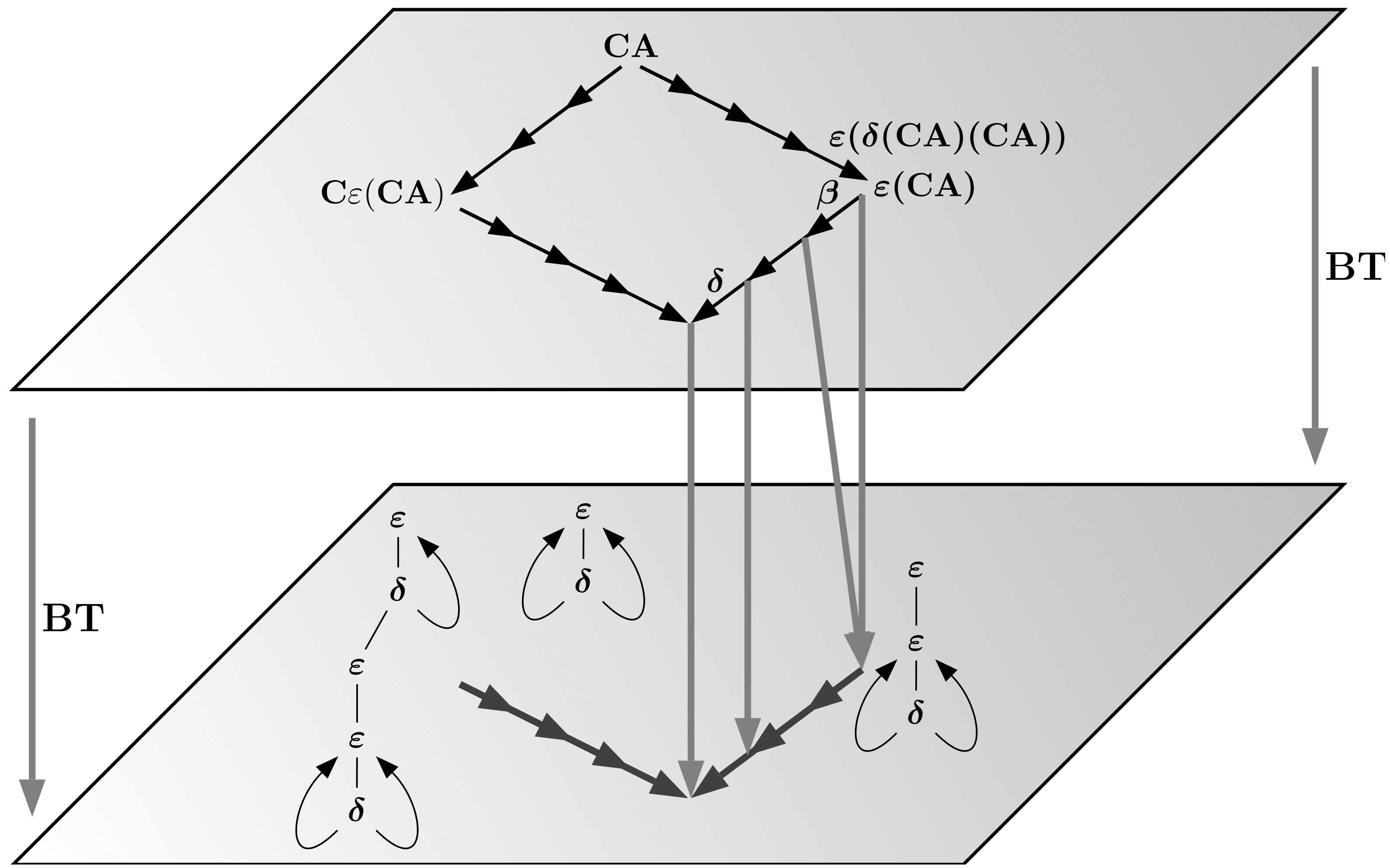
$$\delta xx \to_{\delta_H} x$$

$$Cx \twoheadrightarrow \varepsilon(\delta x(Cx))$$

$$A \twoheadrightarrow CA$$

$$A \twoheadrightarrow CA \twoheadrightarrow \varepsilon(\delta A(CA)) \twoheadrightarrow \varepsilon(\delta(CA)(CA)) \to \varepsilon(CA)$$

$$CA \twoheadrightarrow C(\varepsilon(CA))$$



Question: what about $\lambda^\infty\beta\delta$ and $\lambda^\infty\beta\pi$?

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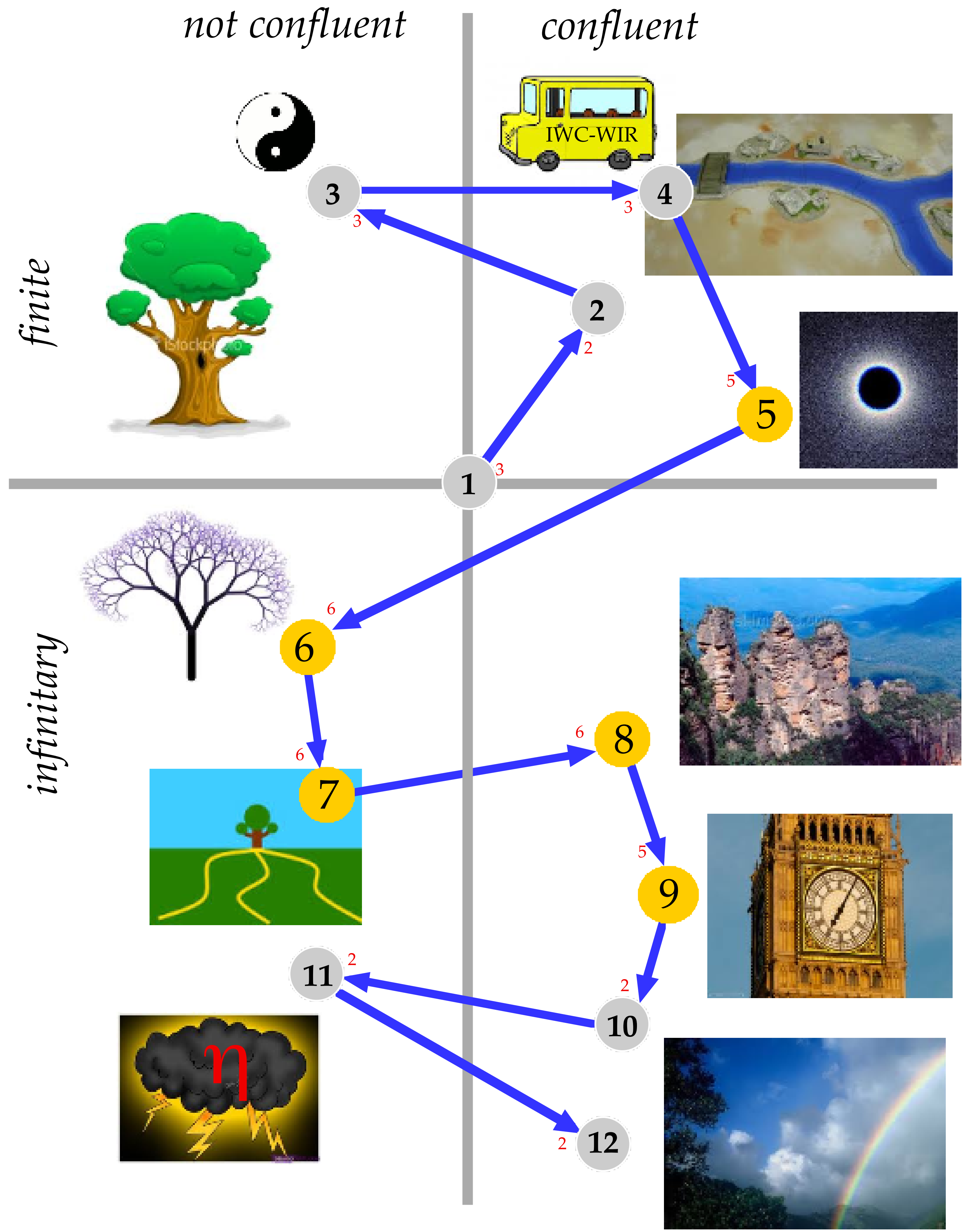

Aczel 1978

coherent contraction schemes

Best wishes
Peter

A General Church-Rosser Theorem

Peter Aczel

We prove the Church-Rosser theorem in a general framework. Our result easily yields the standard result for the lambda calculus but also has wider application.

§1. The Main Theorem

1.1 An expression system consists of an infinite set of variables and a set of forms. Each form has an arity i.e. a finite sequence $k_1, \ldots, k_m$ $(m \geq 0)$ of natural numbers. If $m = 0$ the form is a constant. If $k_1 = \ldots = k_m = 0$ the form is simple.

Expressions are inductively generated using the two rules:-

1) Every variable is an expression.

2) If F is a form with arity $k_1, \ldots, k_m$ $(m \geq 0)$ and $a_1, \ldots, a_m$ are expressions then $F((\vec{x}_1)a_1, \ldots, (\vec{x}_m)a_m)$ is an expression, where for $i = 1, \ldots, m$ $\vec{x}_i$ is a list of k_i variables.

The expression generated by 2) is said to have form F and parts $a_1, \ldots, a_m$. Free and bound occurrences of variables are defined in the usual way, so that occurrences of a variable in the list $\vec{x}_i$ that are free in a_i become bound in $F((\vec{x}_1)a_1, \ldots, (\vec{x}_m)a_m)$. Alphabetic variants of expressions are identified in the standard way.

Below we shall usually write $F(a_1, \ldots, a_m)$ instead of $F((\vec{x}_1)a_1, \ldots, (\vec{x}_m)a_m)$. It must be kept in mind that with this abuse of notation a variable that is free in a_i can become bound in $F(a_1, \ldots, a_m)$. Also, an expression $F(a_1, \ldots, a_m)$ may be the same as an expression $F(b_1, \ldots, b_m)$ while a_i is not the same as b_i for $i = 1, \ldots, m$.

1.2 We shall be concerned with a partial function on the expressions which we shall call a contraction operation. An expression in the domain of the operation will be called a redex and its value under the operation will be called the contraction of the redex. We shall insist that no variable is a redex. Each contraction operation generates a relation of definitional equality.

The methods to prove confluence of orthogonal higher-order rewriting systems both can be adapted to the case where critical pairs are allowed, but only if they are of the form (s, s). Such a critical pair is said to be *trivial*. The notion of trivial critical pair is used to define the class of weakly orthogonal higher-order rewriting systems; the definition is analogous to the one for the first-order case.

Definition 3. *A higher-order rewriting system is* weakly orthogonal *if it is left-linear and all its critical pairs are trivial.*

Examples of weakly orthogonal rewriting systems that are not orthogonal are $\{a \to b, f(a) \to f(b)\}$ and $\{f(x) \to f(b), f(a) \to f(b)\}$. Moreover, lambda-calculus with both β-and η-reduction is a weakly orthogonal rewriting system.

Typed Lambda Calculi and Applications
Lecture Notes in Computer Science Volume 664, 1993, pp 306-317

Orthogonal higher-order rewrite systems are confluent

Abstract

The results about higher-order critical pairs and the confluence of OHRSs provide a firm foundation for the further study of higher-order rewrite systems. It should now be interesting to lift more results and techniques both from term-rewriting and λ-calculus to the level of HRSs. For example termination proof techniques are much studied for TRSs and are urgently needed for HRSs; similarly the extension of our result to weakly orthogonal HRSs or even to Huet's "parallel closed" systems is highly desirable. Conversely, a large body of λ-calculus reduction theory has been lifted to CRSs [10] already and should be easy to carry over to HRSs.

Finally there is the need to extend the notion of an HRS to more general left-hand sides. For example the *eta*-rule for the *case*-construct on disjoint unions [15] *case*(U,λx.F(inl(*x*)),λy.G(inr(y))) → F(U) is outside our framework, whichever way it is oriented.

Van Oostrom, van Raamsdonk 1994

Weak Orthogonality Implies Confluence: the Higher-Order Case
Vincent van Oostrom and Femke van Raamsdonk
Technical Report: ISRL-94-5
December, 1994

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ A\end{array}}{A \vee B} \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C} \quad \to \quad \begin{array}{c}\vdots\\ A\\ \vdots\\ C\end{array} \qquad\qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ B\end{array}}{A \vee B} \qquad \begin{array}{c}[A]\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]\\ \vdots\\ C\end{array}}{C} \quad \to \quad \begin{array}{c}\vdots\\ B\\ \vdots\\ C\end{array}$$

These rules can be written in the formalism of Combinatory Reduction Systems. They then take the following form:

$$\begin{array}{lcl}\mathtt{el}(\mathtt{inl}(Z),[x]Z_0(x),[y]Z_1(y)) & \to & Z_0(Z)\\ \mathtt{el}(\mathtt{inr}(Z),[x]Z_0(x),[y]Z_1(y)) & \to & Z_1(Z)\end{array}$$

$\lambda\beta\eta \vDash \mathrm{CR}$

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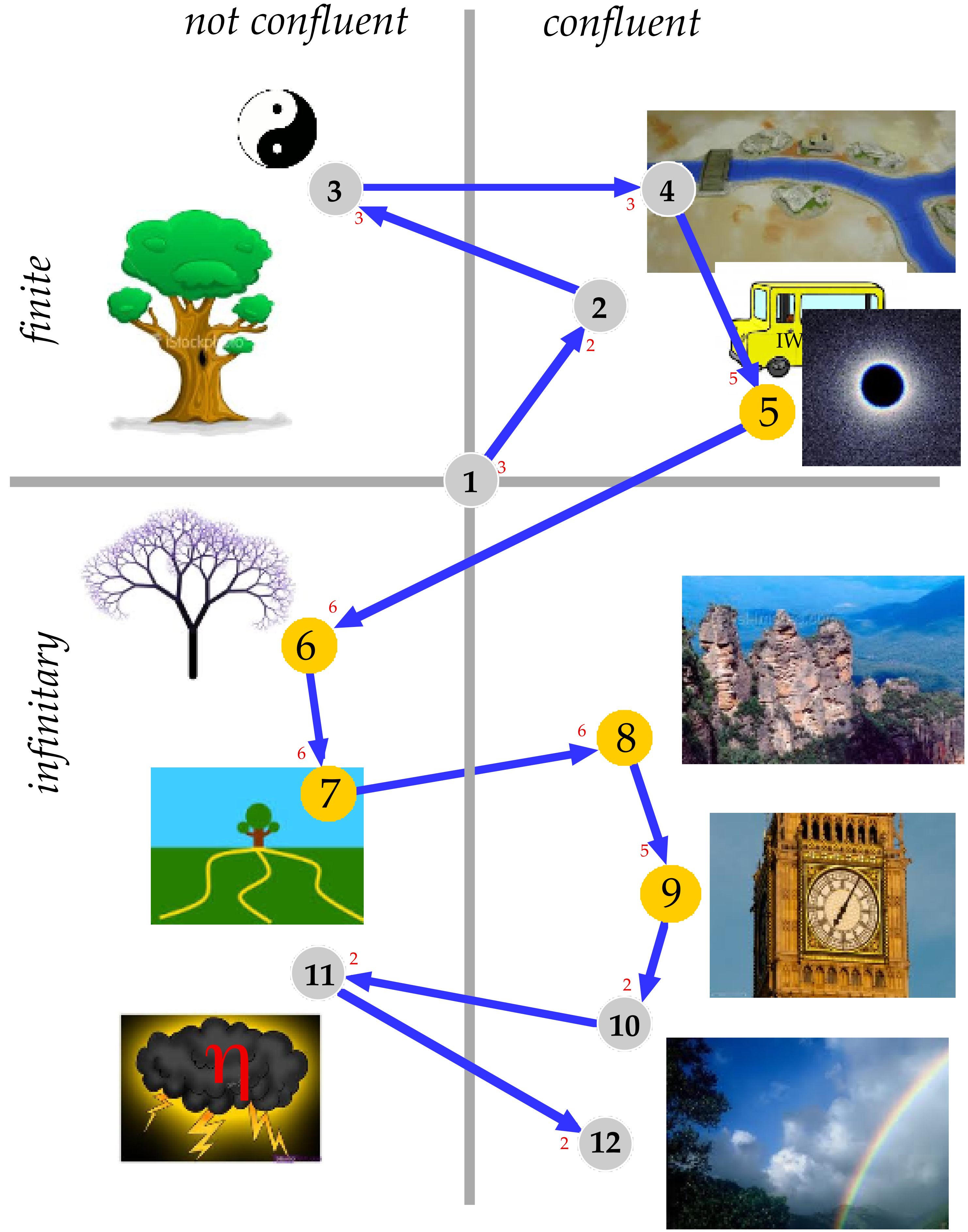
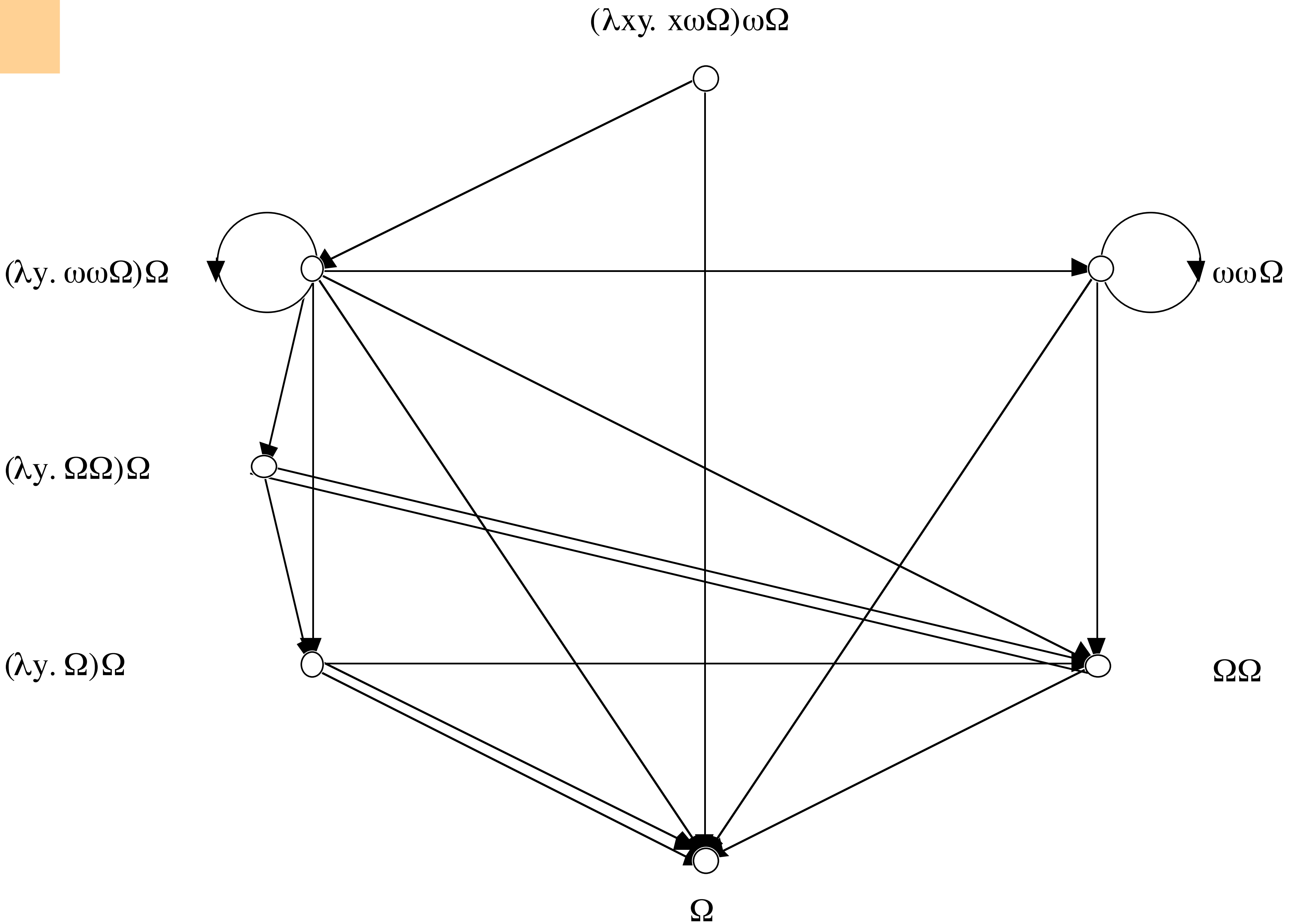


Modulo unsolvables:

$(\lambda x.Z(x))Z' \to Z(Z')$	(β)
$M \to \Omega$	if $M \neq \Omega$ is unsolvable (uns)
$\Omega M \to \Omega$	(Ω_l)
$\lambda x.\Omega \to \Omega$	(Ω_d)

$\lambda\beta\Omega \vDash \mathrm{CR}$

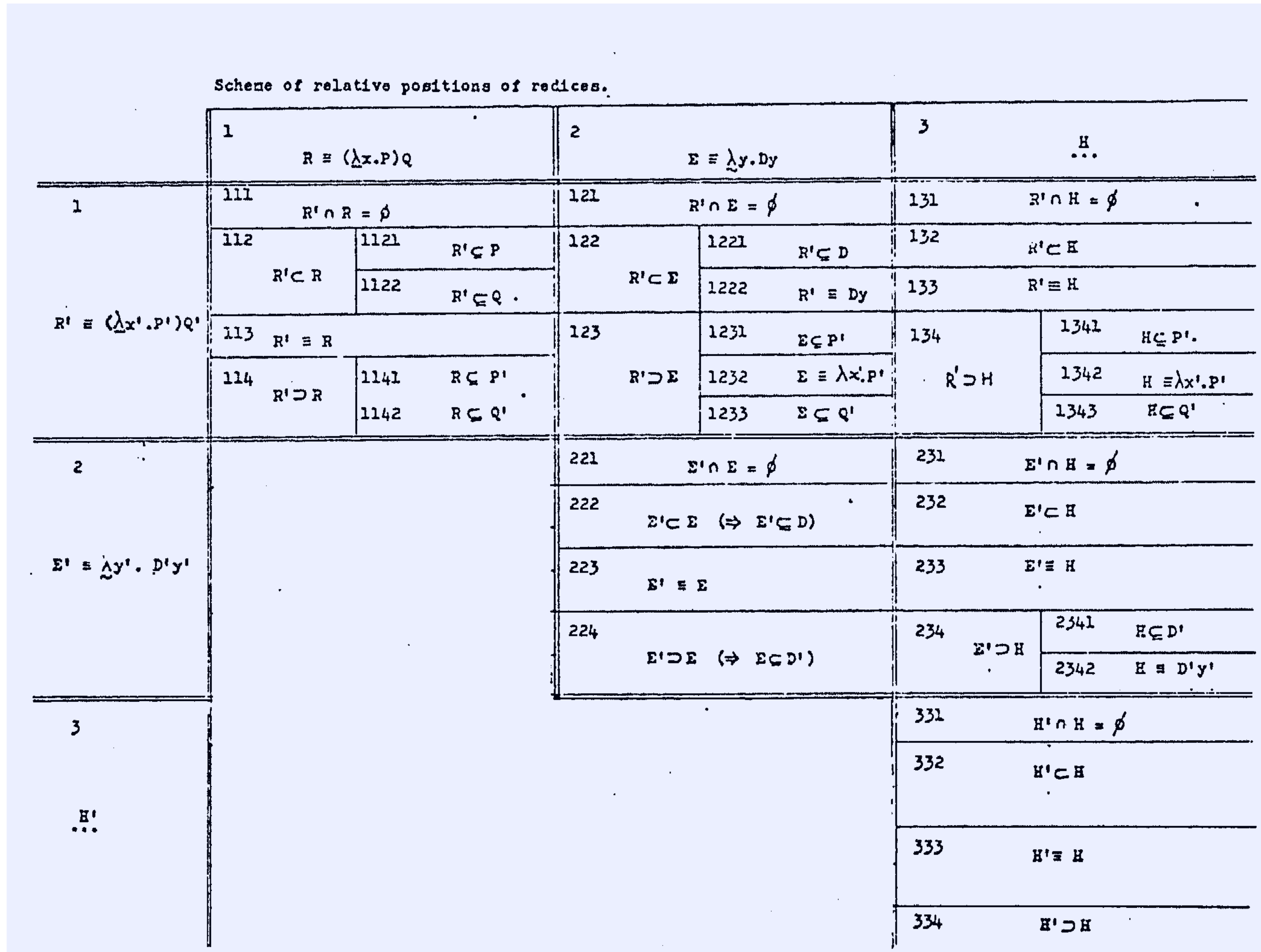


Blue preprint 1976, Barendregt, Bergstra, Klop, Volken: youth sentiment and contortuous casuistics

$\lambda\beta\eta\Omega \vDash CR$

later question:

$\lambda^{\infty}\beta\eta\Omega \vDash CR^{\infty}$

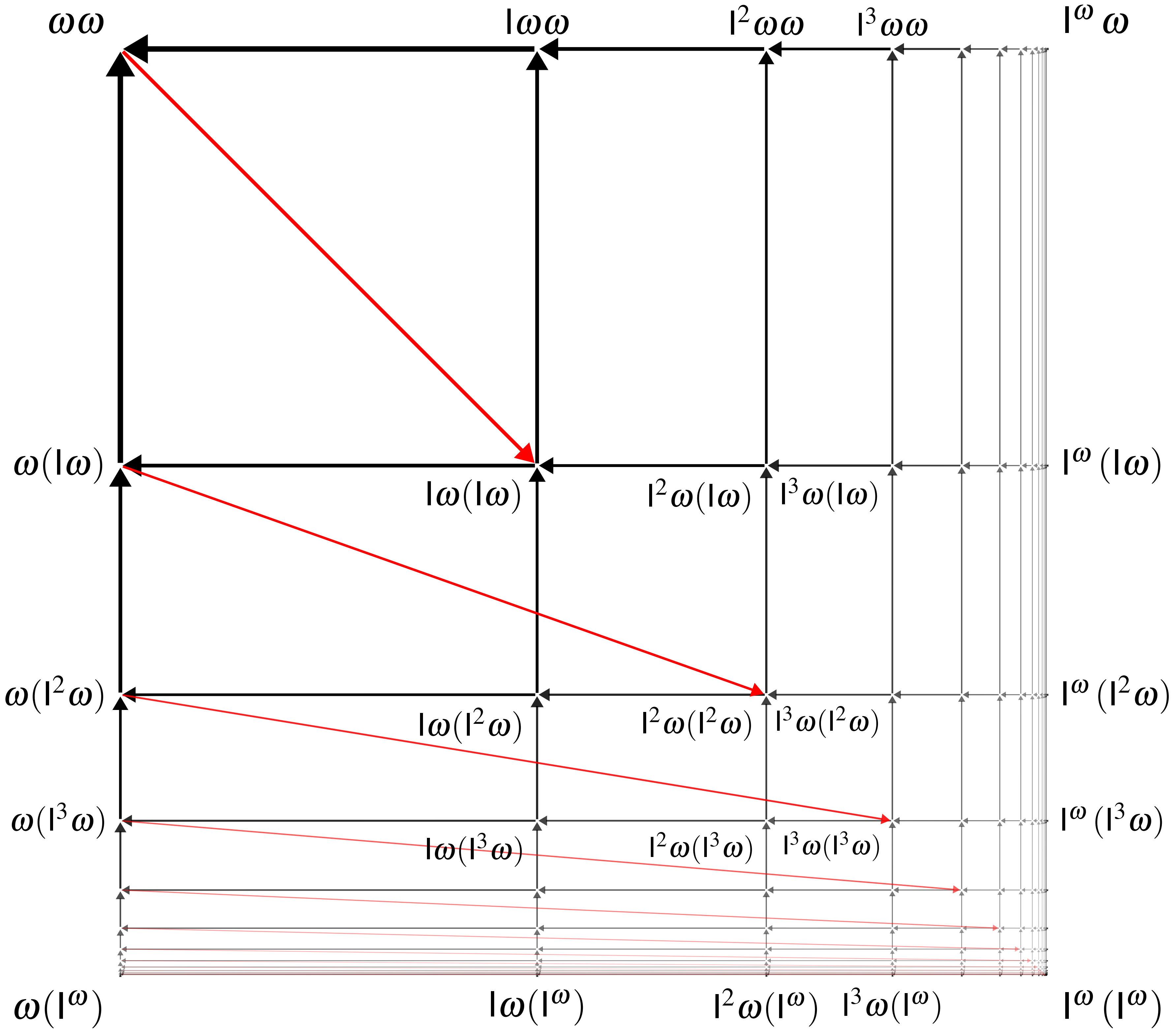


Scheme of relative positions of redices.

	1 $R \equiv (\lambda x.P)Q$	2 $E \equiv \lambda y.Dy$	3 H
1 $R' \equiv (\lambda x'.P')Q'$	111 $R' \cap R = \emptyset$ 112 $R' \subset R$: 1121 $R' \subsetneq P$; 1122 $R' \subsetneq Q$ 113 $R' \equiv R$ 114 $R' \supset R$: 1141 $R \subsetneq P'$; 1142 $R \subsetneq Q'$	121 $R' \cap E = \emptyset$ 122 $R' \subset E$: 1221 $R' \subsetneq D$; 1222 $R' \equiv Dy$ 123 $R' \supset E$: 1231 $E \subsetneq P'$; 1232 $E \equiv \lambda x'.P'$; 1233 $E \subsetneq Q'$	131 $R' \cap H = \emptyset$ 132 $R' \subset H$ 133 $R' \equiv H$ 134 $R' \supset H$: 1341 $H \subsetneq P'$; 1342 $H \equiv \lambda x'.P'$; 1343 $H \subsetneq Q'$
2 $E' \equiv \lambda y'.D'y'$		221 $E' \cap E = \emptyset$ 222 $E' \subset E$ ($\Rightarrow E' \subsetneq D$) 223 $E' \equiv E$ 224 $E' \supset E$ ($\Rightarrow E \subsetneq D'$)	231 $E' \cap H = \emptyset$ 232 $E' \subset H$ 233 $E' \equiv H$ 234 $E' \supset H$: 2341 $H \subsetneq D'$; 2342 $H \equiv D'y'$
3 H'			331 $H' \cap H = \emptyset$ 332 $H' \subset H$ 333 $H' \equiv H$ 334 $H' \supset H$

In Barendregt 84: section 15.2, 8 pages

Werkweek λ-calculus in de molen te Varik
juni 75.



ωω
Iωω
I²ωω
I³ωω
I^ω ω
ω(Iω)
Iω(Iω)
I²ω(Iω)
I³ω(Iω)
I^ω (Iω)
ω(I²ω)
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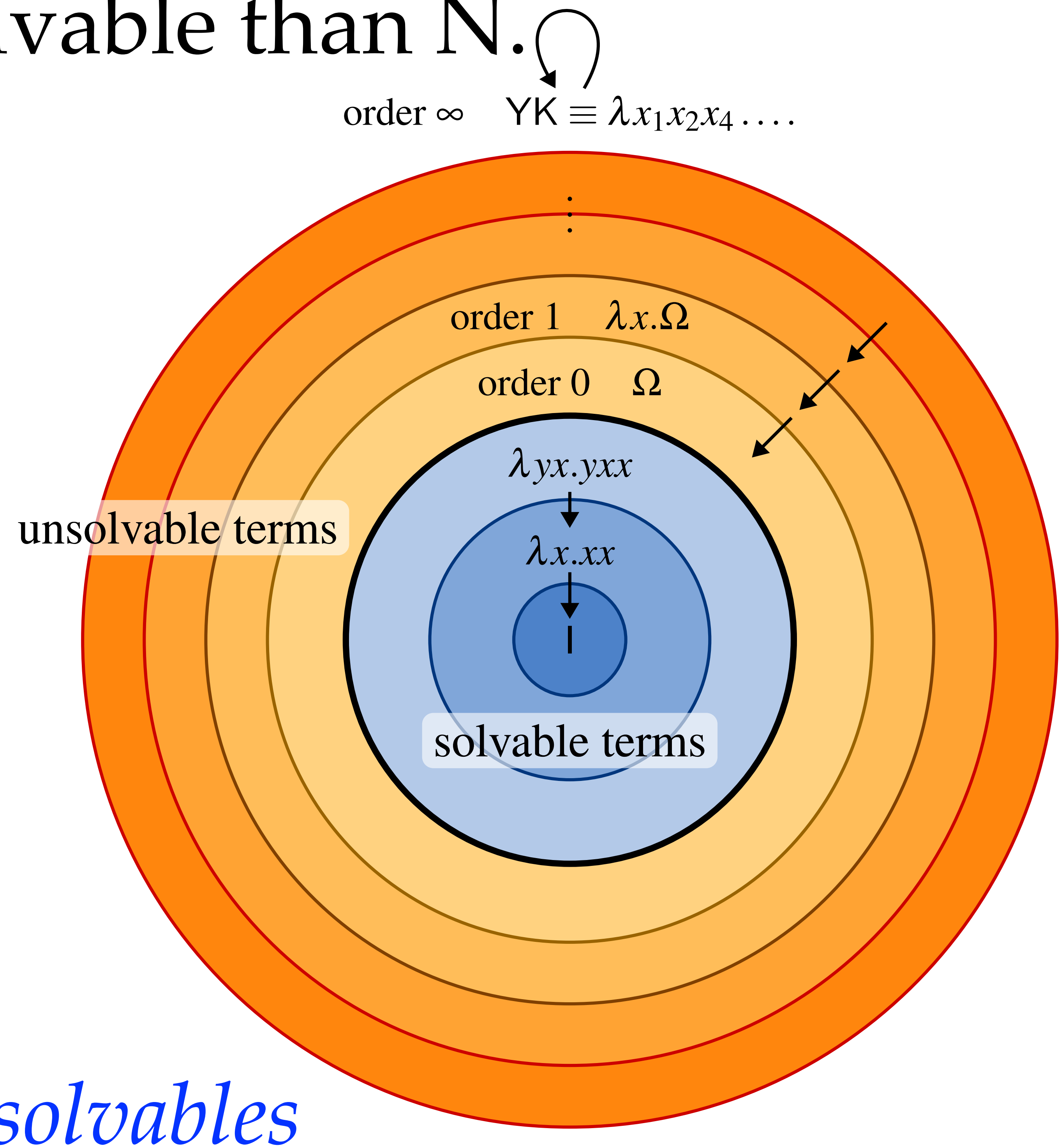
Iω(I^ω)
I²ω(I^ω)
I³ω(I^ω)
I^ω (I^ω)

Statman 1978

instead of $MA \twoheadrightarrow_\beta N$, write $M \overset{A}{\Rightarrow} N$

$M \Rightarrow N$: M is more solvable than N.



Every countable poset is embeddable in poset of unsolvables

head normalization theorems

$\Box\,\mathsf{root}$

$\supseteq$

Ω_{BeT} (mute terms, no root stable form) $= \Box\Diamond\,\mathsf{root}$

$\supseteq$

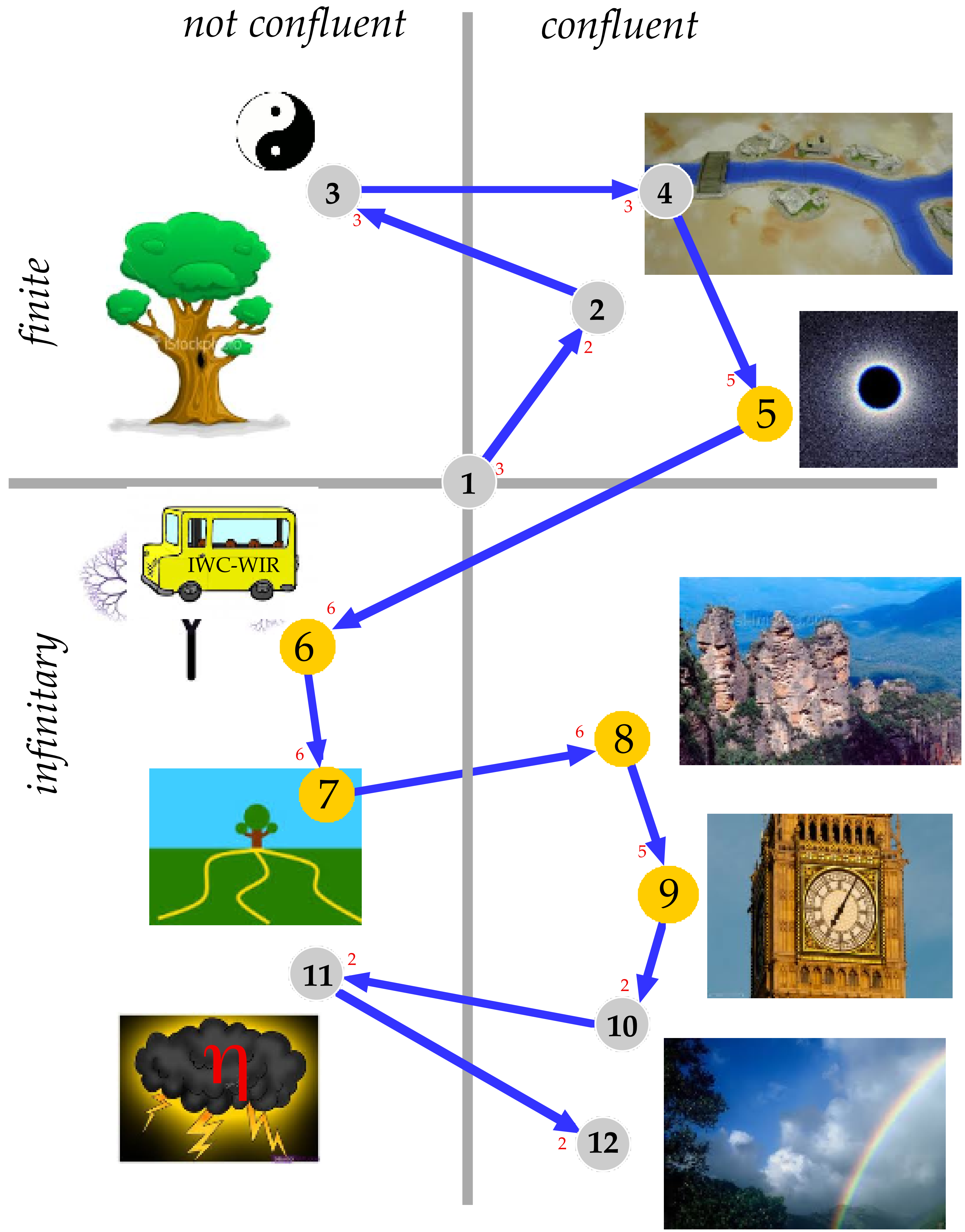
Ω_{LLT} (no weak head normal form) $= \Box\Diamond\,\mathsf{lazy} = \Box\,\mathsf{lazy}$

$\supseteq$

Ω_{BT} (no head normal form, unsolvables) $= \Box\Diamond\,\mathsf{head} = \Box\,\mathsf{head}$

$\Box\Diamond\,\mathsf{head} = \Box\Diamond\,\mathsf{spine}$, $\Box\,\mathsf{head} = \Box\,\mathsf{spine}$

$\Box\Diamond\,\mathsf{spine} = \Box\,\mathsf{spine}$


not confluent
confluent
finite
infinitary
1
2
3
4
5
6
7
8
9
10
11
12
IWC-WIR
η

REWRITE, REWRITE, REWRITE, REWRITE, REWRITE, ...*

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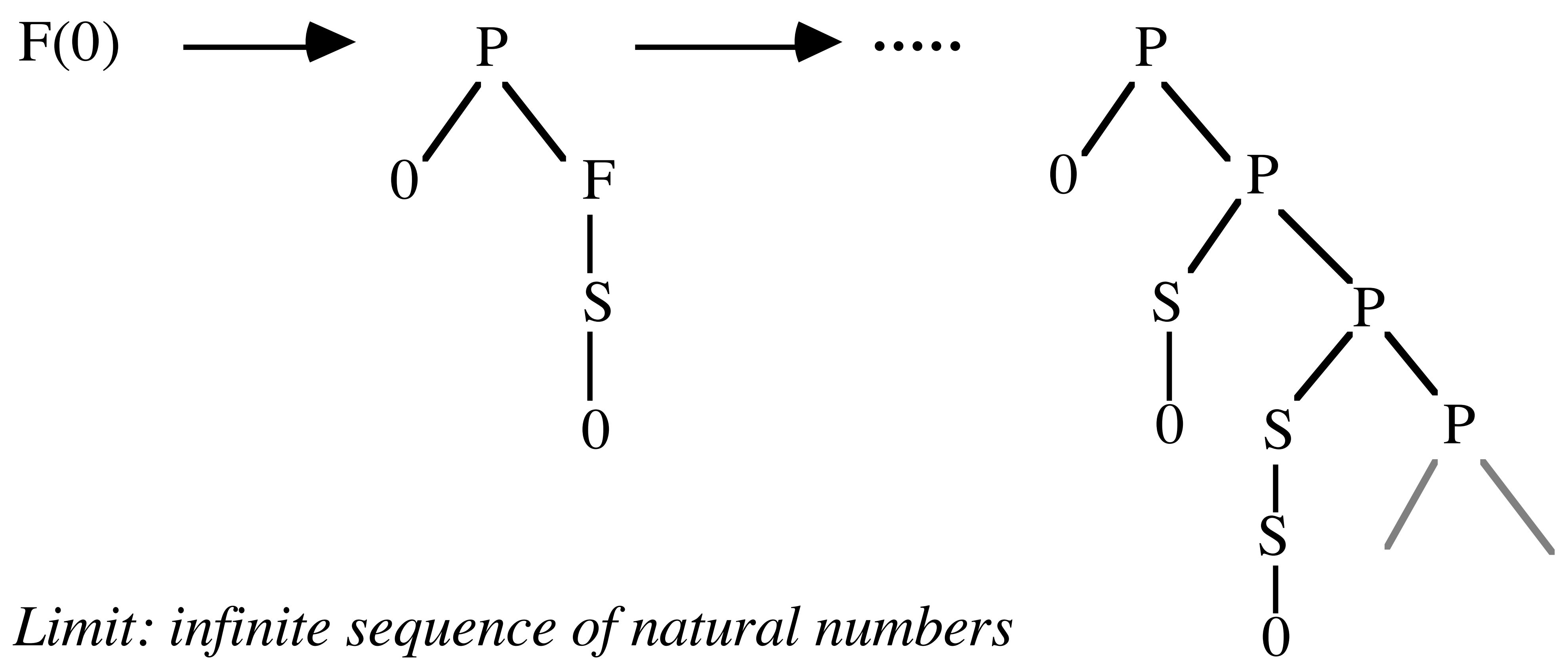
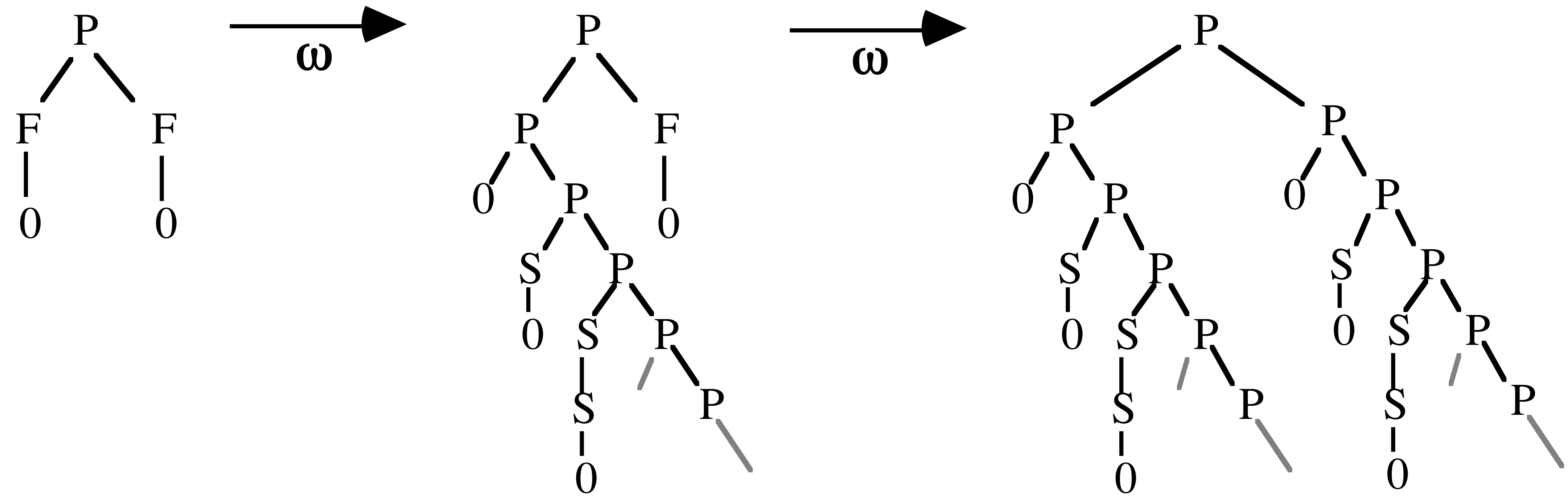
Abstract. We study properties of rewrite systems that are not necessarily terminating, but allow instead for transfinite derivations that have a limit. In particular, we give conditions for the existence of a limit and for its uniqueness and relate the operational and algebraic semantics of infinitary theories. We also consider sufficient completeness of hierarchical systems.

Is there no limit?
—Job 16:3



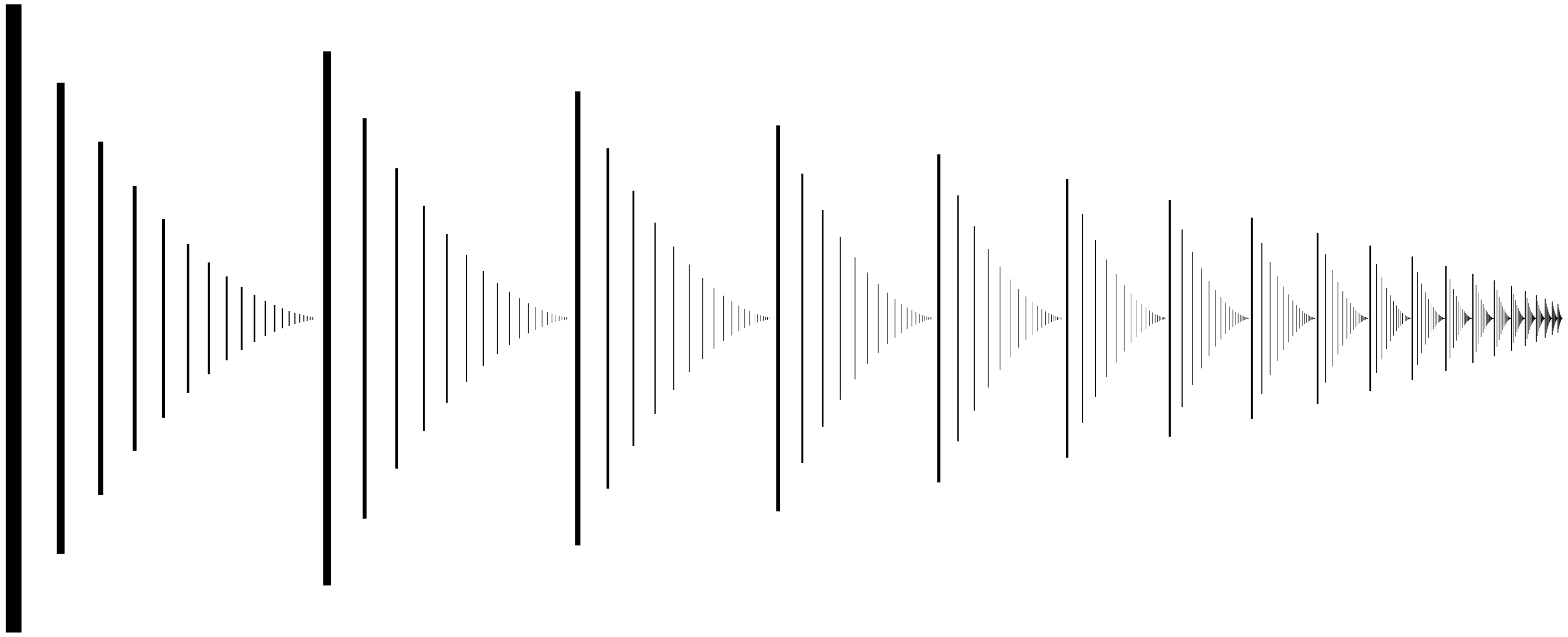
Limit: infinite sequence of natural numbers

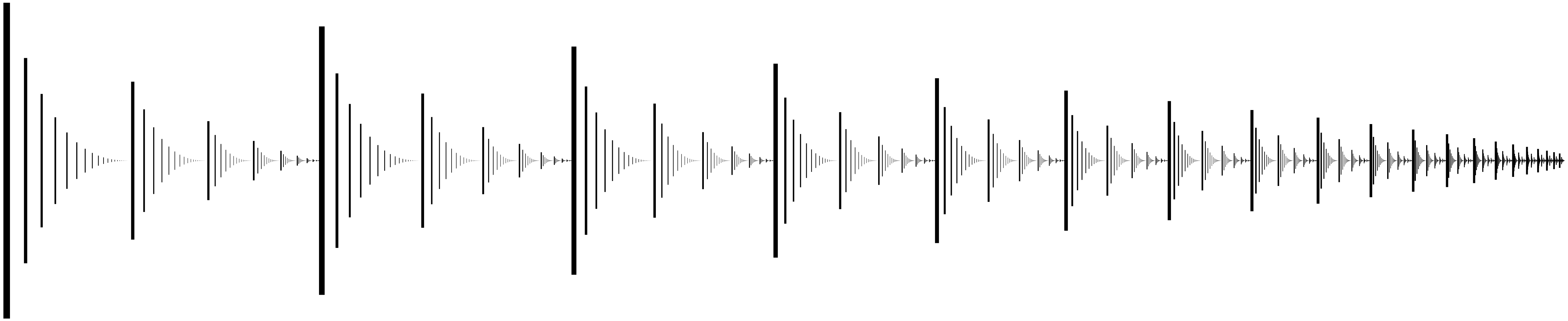
$$F(x) \rightarrow P(x, F(S(x)))$$

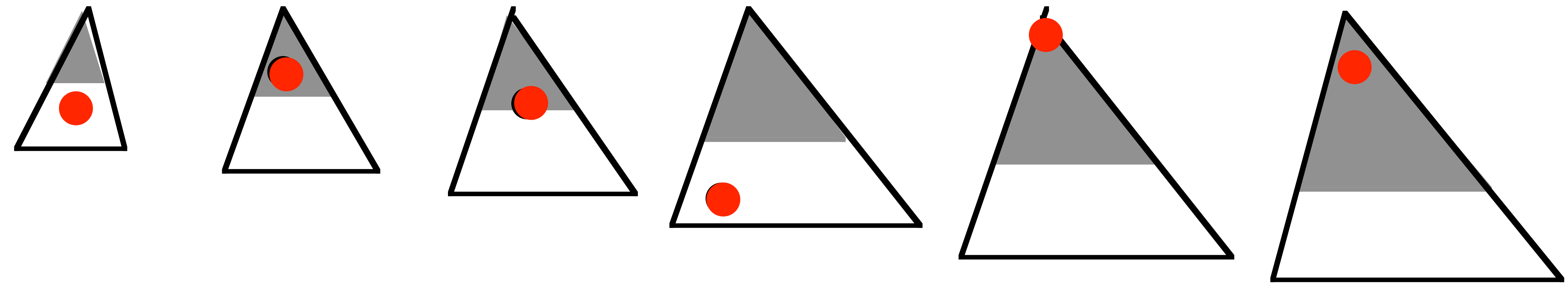


Transfinite reduction sequence of length $\omega + \omega$

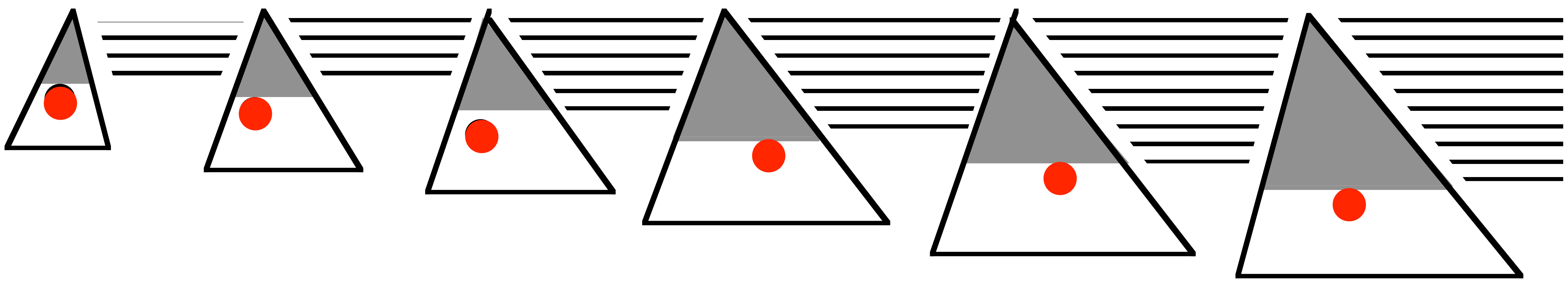
$$F(x) \rightarrow P(x, F(S(x)))$$







Cauchy converging reduction sequence: activity may occur everywhere

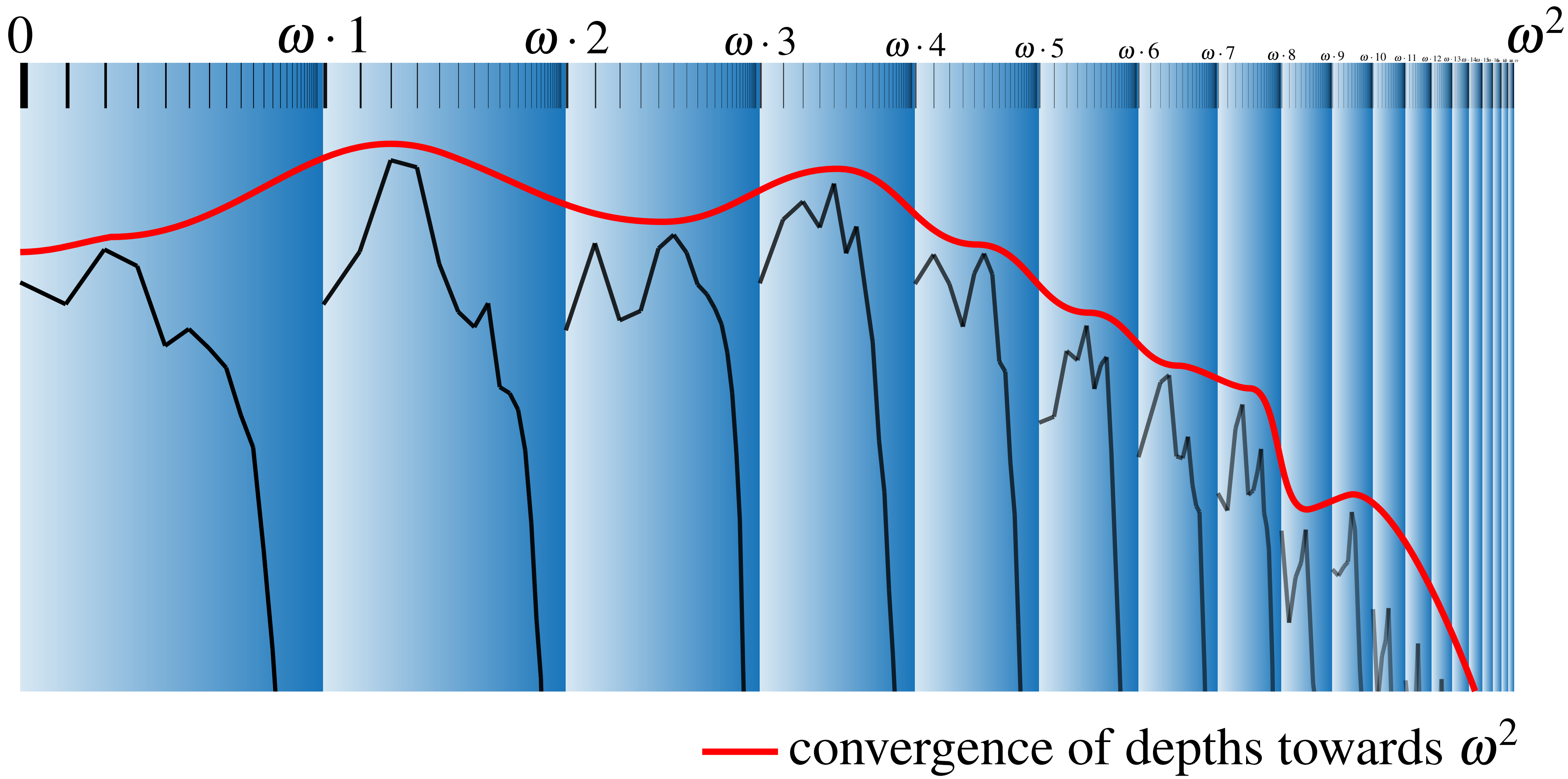


Strongly converging reduction sequence, with descendant relations

difference between CC and SC: looping terms

Kennaway-de Vries 1992; De Vrijer, Grabmayer, Endrullis, Hendriks, Simonsen 2012

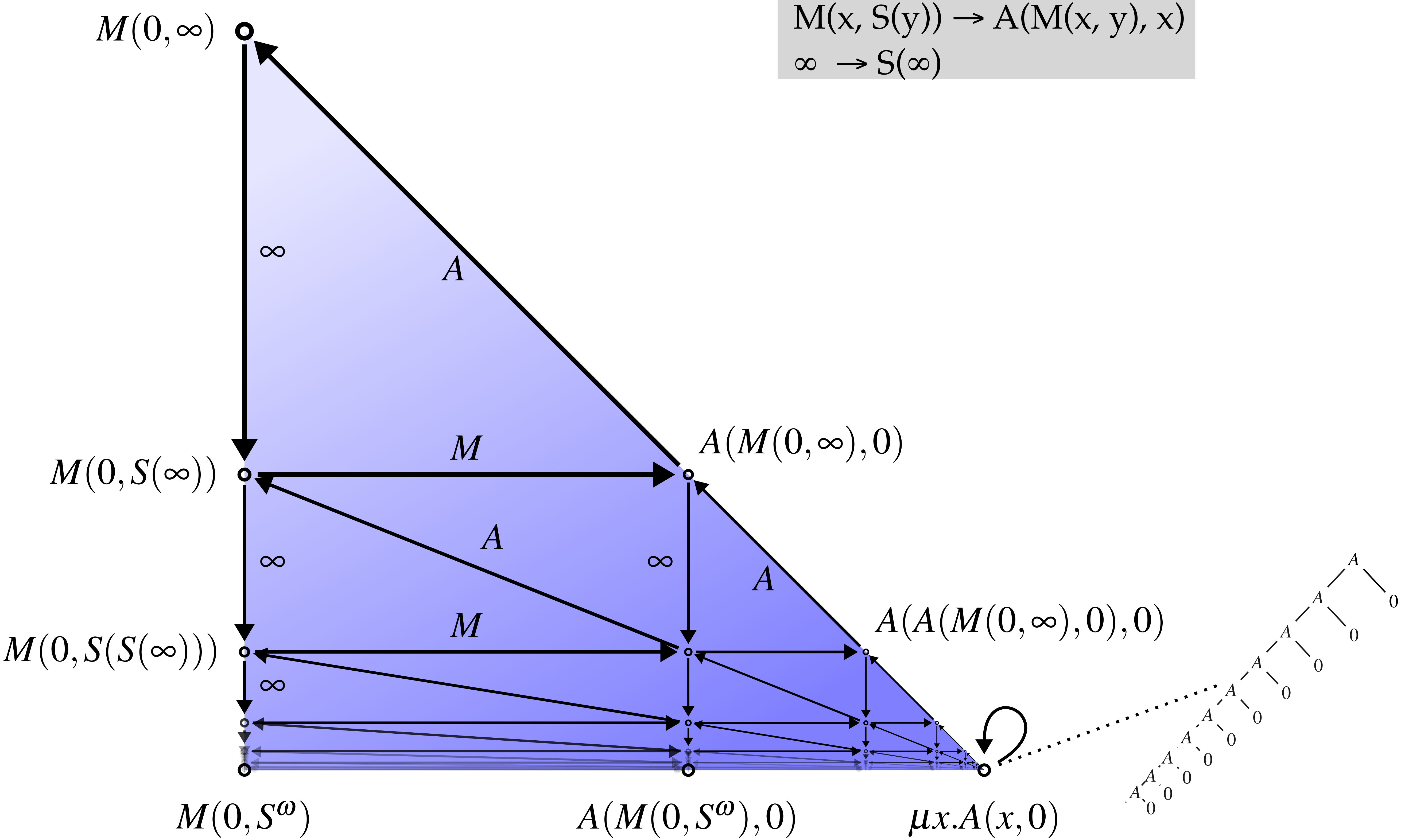
strong convergence: redex depth to infinity

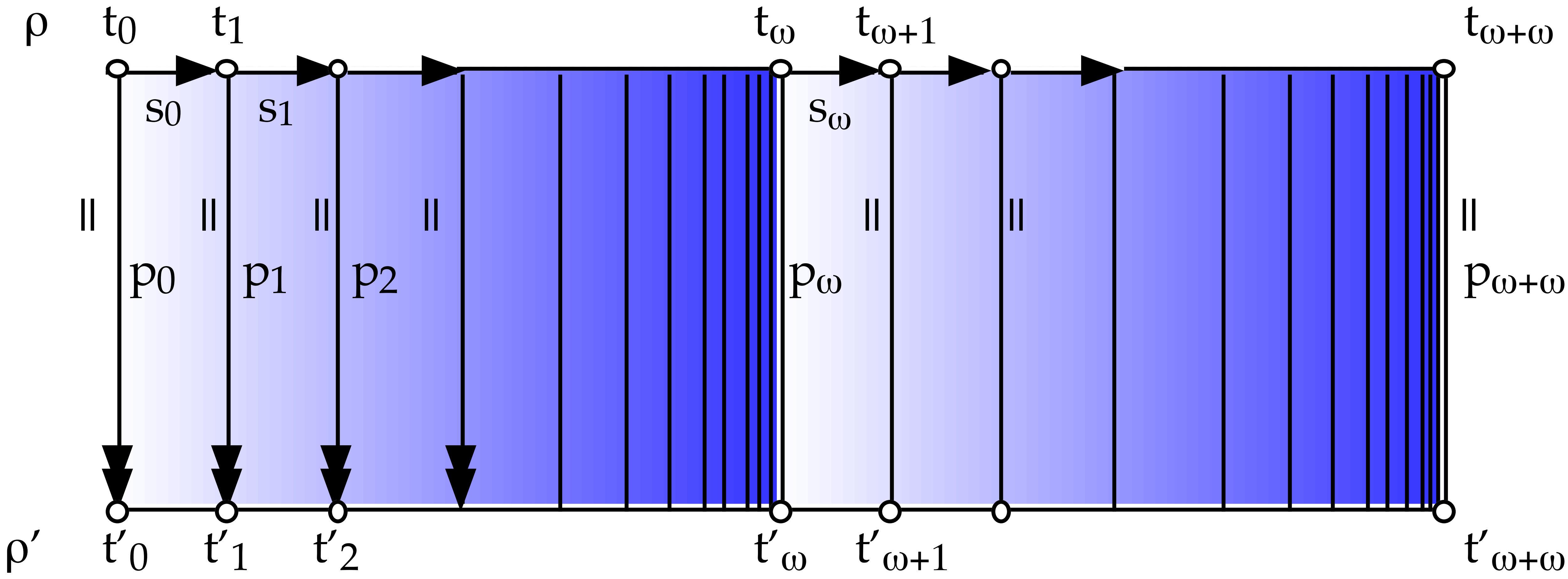


Finitary rewriting	**Infinitary or transfinite rewriting**
finite reduction	strongly convergent reduction
infinite reduction	divergent reduction ("stagnating")
normal form	possibly infinite normal form
CR: two coinitial finite reductions can be prolonged to a common term	CR^{∞}: two coinitial strongly convergent reductions can be prolonged by strongly convergent reductions to a common term
UN: two coinitial reductions ending in normals forms, end in the same normal form	UN^{∞}: two coinitial strongly convergent reductions ending in (possibly infinite) normal forms, end in the same normal form
SN: all reductions lead eventually to a normal form	SN^{∞}: all reductions lead eventually to a possibly infinite normal form, equivalently: there is no divergent reduction
WN: there is a finite reduction to a normal form	WN^{∞}: there is a strongly convergent reduction to a possibly infinite normal form

zero times infinity

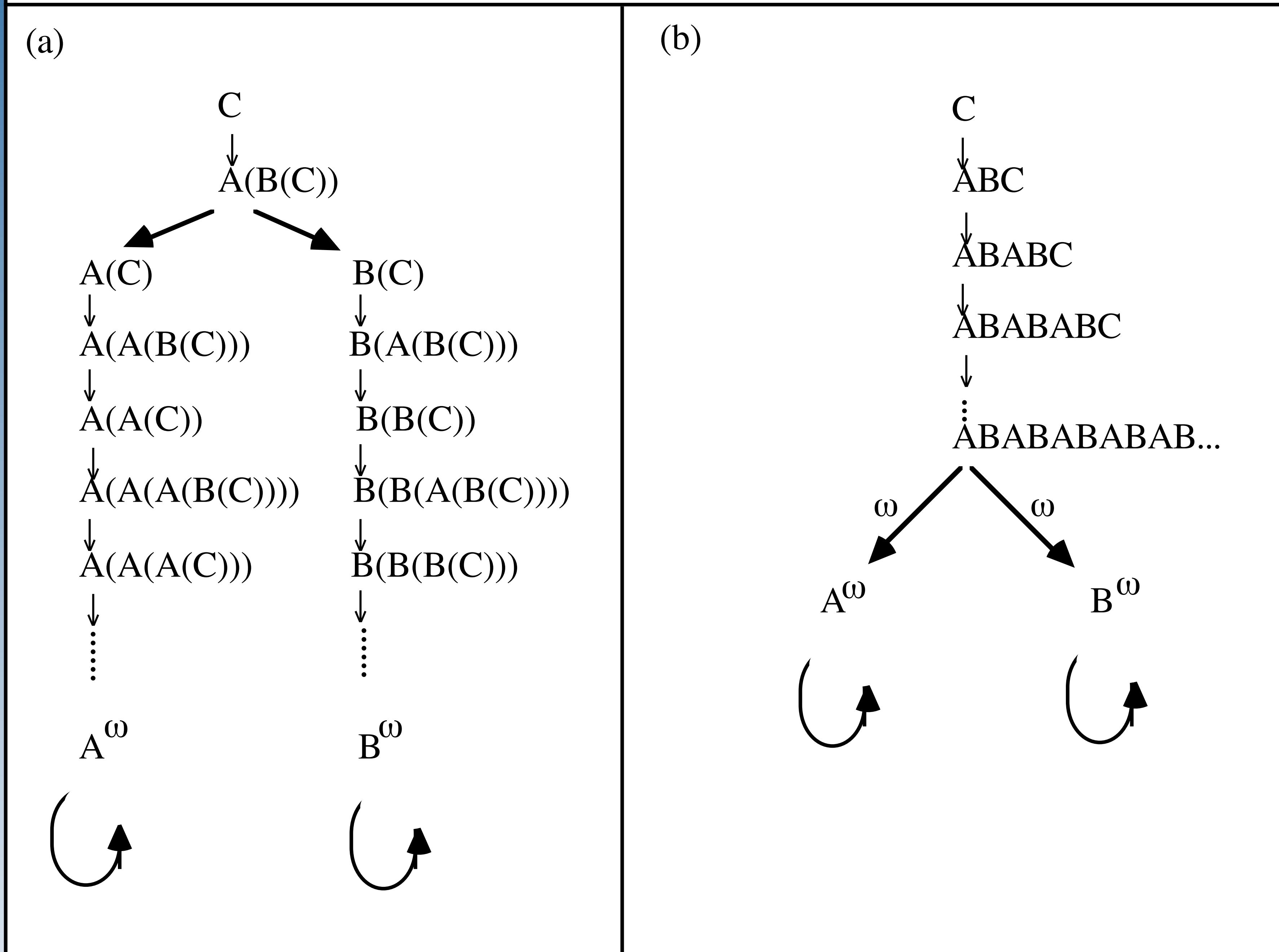
$A(x, 0) \to x$
$A(x, S(y)) \to S(A(x, y))$
$M(x, 0) \to 0$
$M(x, S(y)) \to A(M(x, y), x)$
$\infty \to S(\infty)$




ρ
t0
t1
tω
tω+1
tω+ω
s0
s1
sω
p0
p1
p2
pω
pω+ω
ρ′
t′0
t′1
t′2
t′ω
t′ω+1
t′ω+ω

not CR$^\infty$

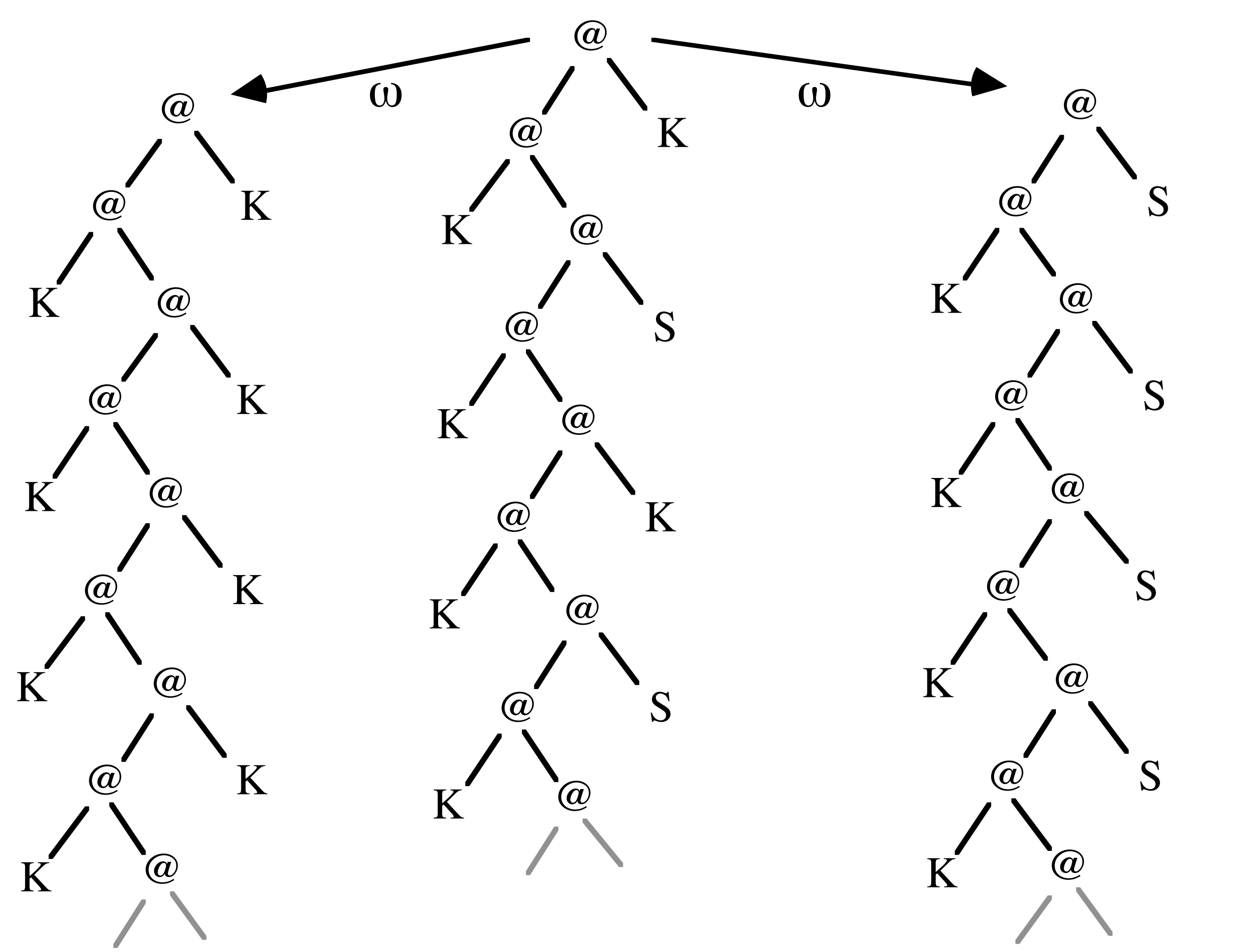
$A(x) \to x$
$B(x) \to x$
$C \to A(B(C))$



Failure of infinitary confluence

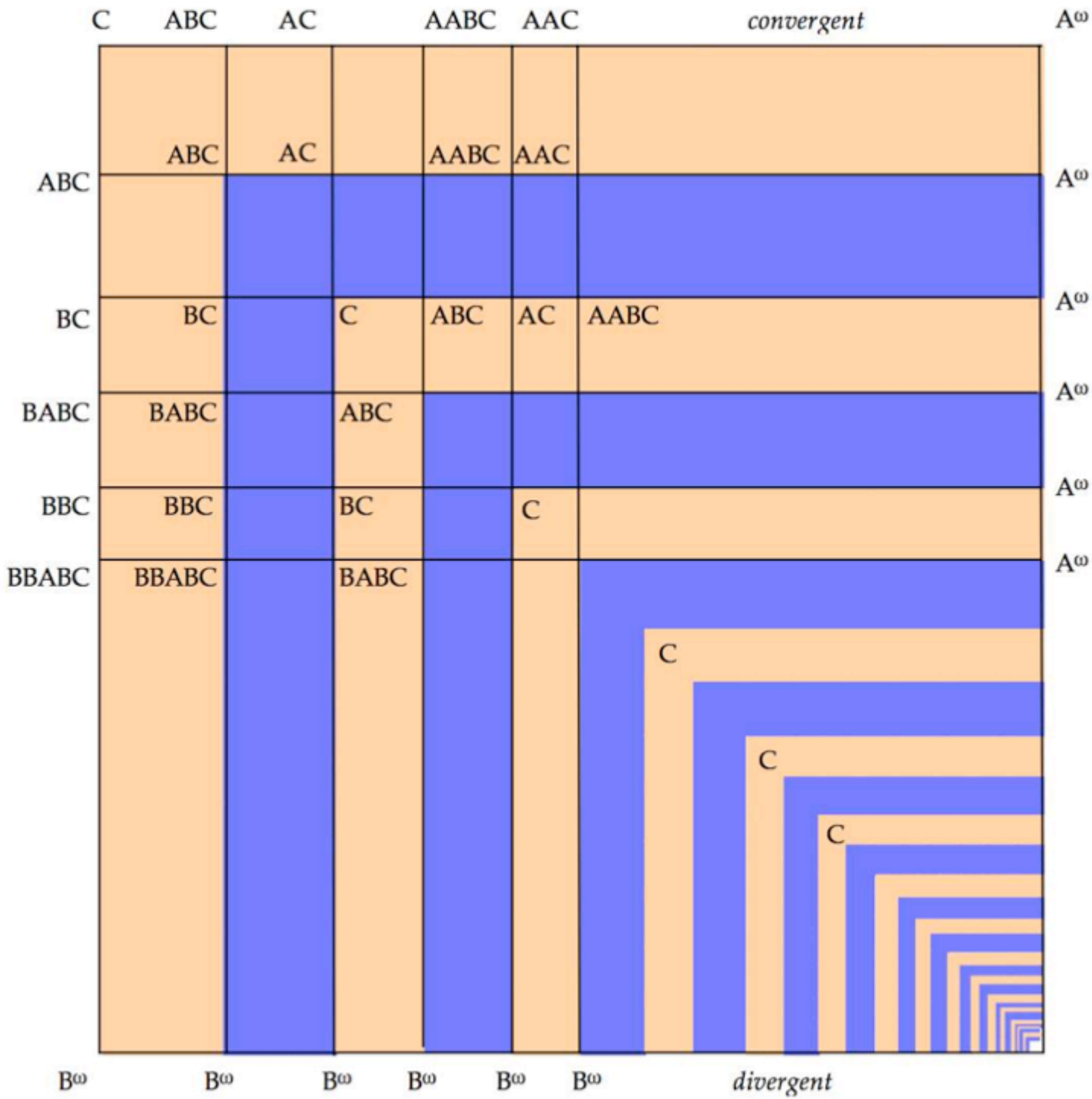
Sxyz → xz(yz)	@(@(@(S, x), y), z) → @(@(x, z), @(y, z))
Kxy → x	@(@(K, x), y) → x

collapsing contexts



Failure of infinitary confluence for Combinatory Logic

Failure of CR^{∞}

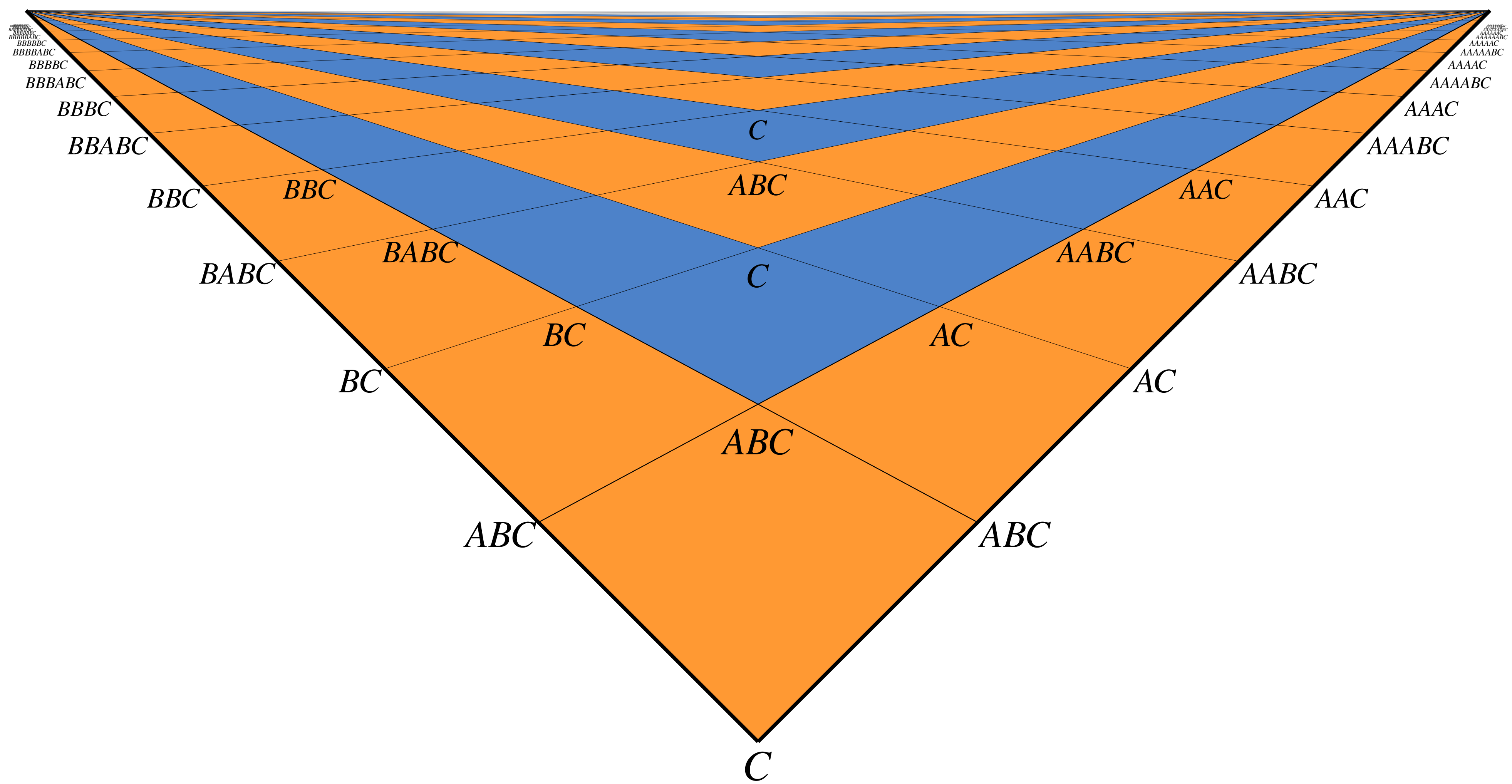


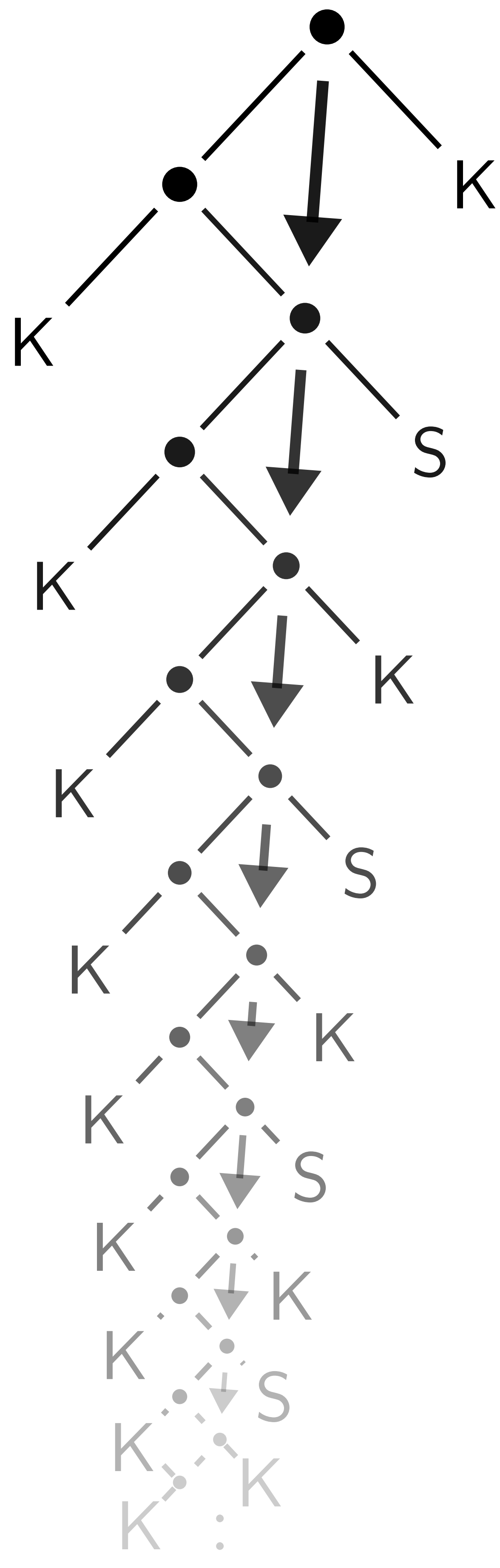
$A(x) \to x$

$B(x) \to x$

$C \to A(B(C))$

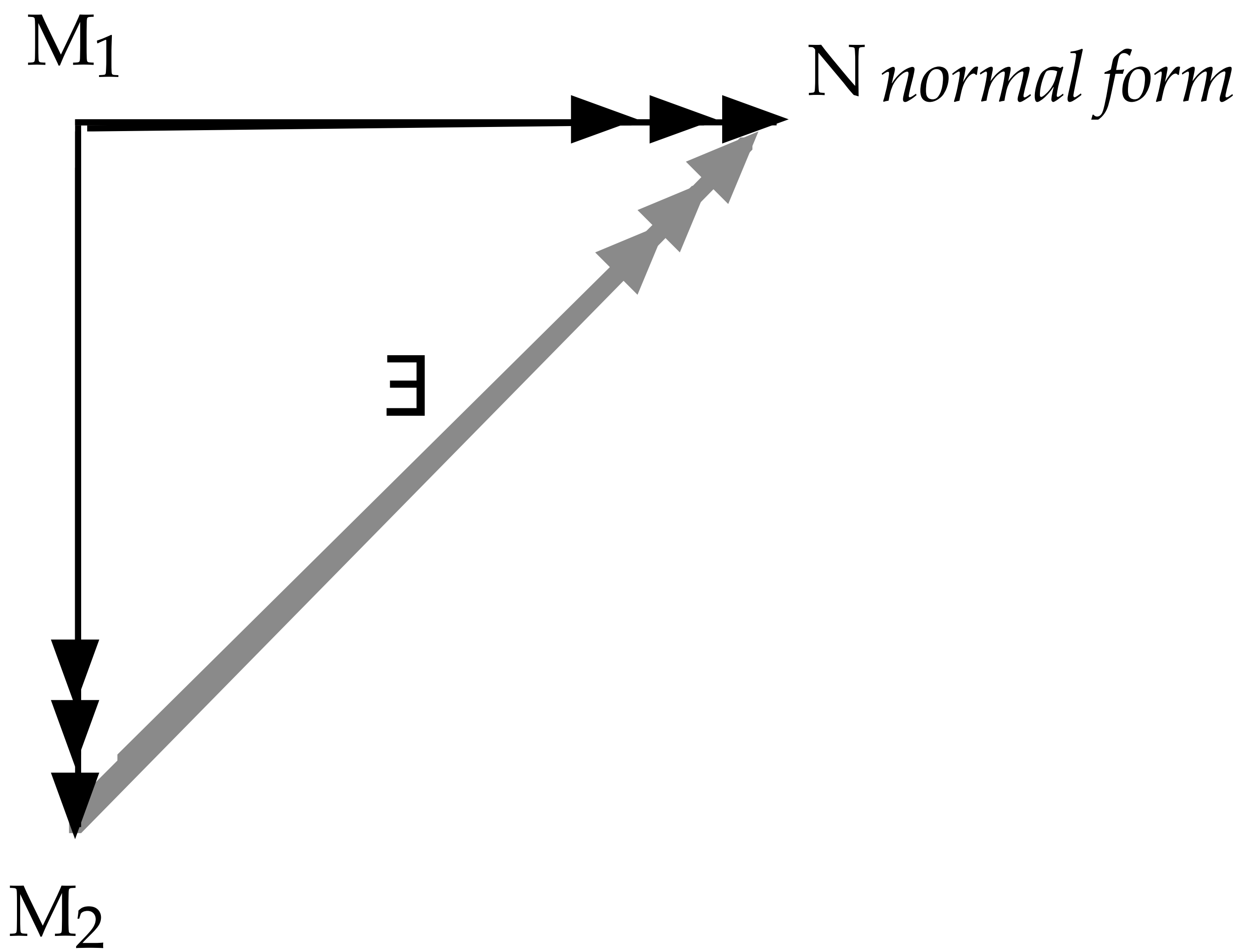
ABC-counterexample in perspective: euclidean distance = tree distance

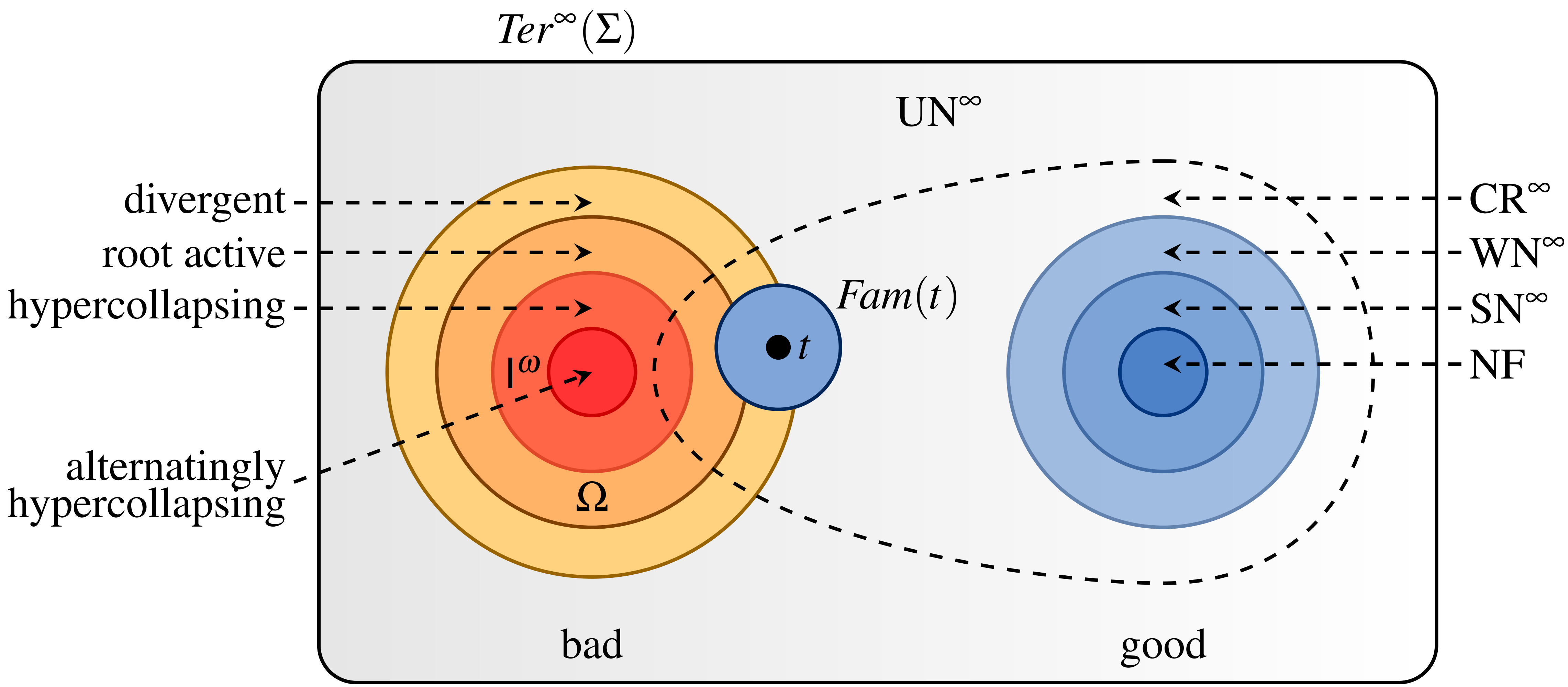




Example 2.4. The '*ABC*-example' that we saw in the preceding example also works in the much more important rewrite system Combinatory Logic CL, with the usual three basic combinators $\mathsf{I}, \mathsf{K}, \mathsf{S}$ and their corresponding reductions rules (see, e.g., Barendregt [2]), and also in infinitary λ-calculus that we will consider in more detail in the next section. The figure on the right, with the infinite collapsing tower of two different *collapsing contexts* $\mathsf{K}\Box\mathsf{K}$ and $\mathsf{K}\Box\mathsf{S}$ shows how the *ABC*-counterexample can be simulated using a fixed-point construction in those calculi. To see that this is indeed a CR^∞-counterexample, note that $\mu x.\mathsf{K}(\mathsf{K}x\mathsf{S})\mathsf{K} \twoheadrightarrow\!\!\!\!\rightarrow \mu x.\mathsf{K}x\mathsf{S}$ and also $\mu x.\mathsf{K}(\mathsf{K}x\mathsf{S})K \twoheadrightarrow\!\!\!\!\rightarrow \mu x.\mathsf{K}x\mathsf{K}$, while $\mu x.\mathsf{K}x\mathsf{S}$ and $\mu x.\mathsf{K}x\mathsf{K}$ only reduce to themselves (in any countable ordinal number of steps, by the way).

Ketema-Simonsen, with UN$^\infty$ *as corollary*





For all terms t in an orthogonal TRS, we have

$$Fam(t) \cap \mathrm{HC} = \varnothing \quad \Rightarrow \quad \mathrm{CR}^{\infty}(t)$$

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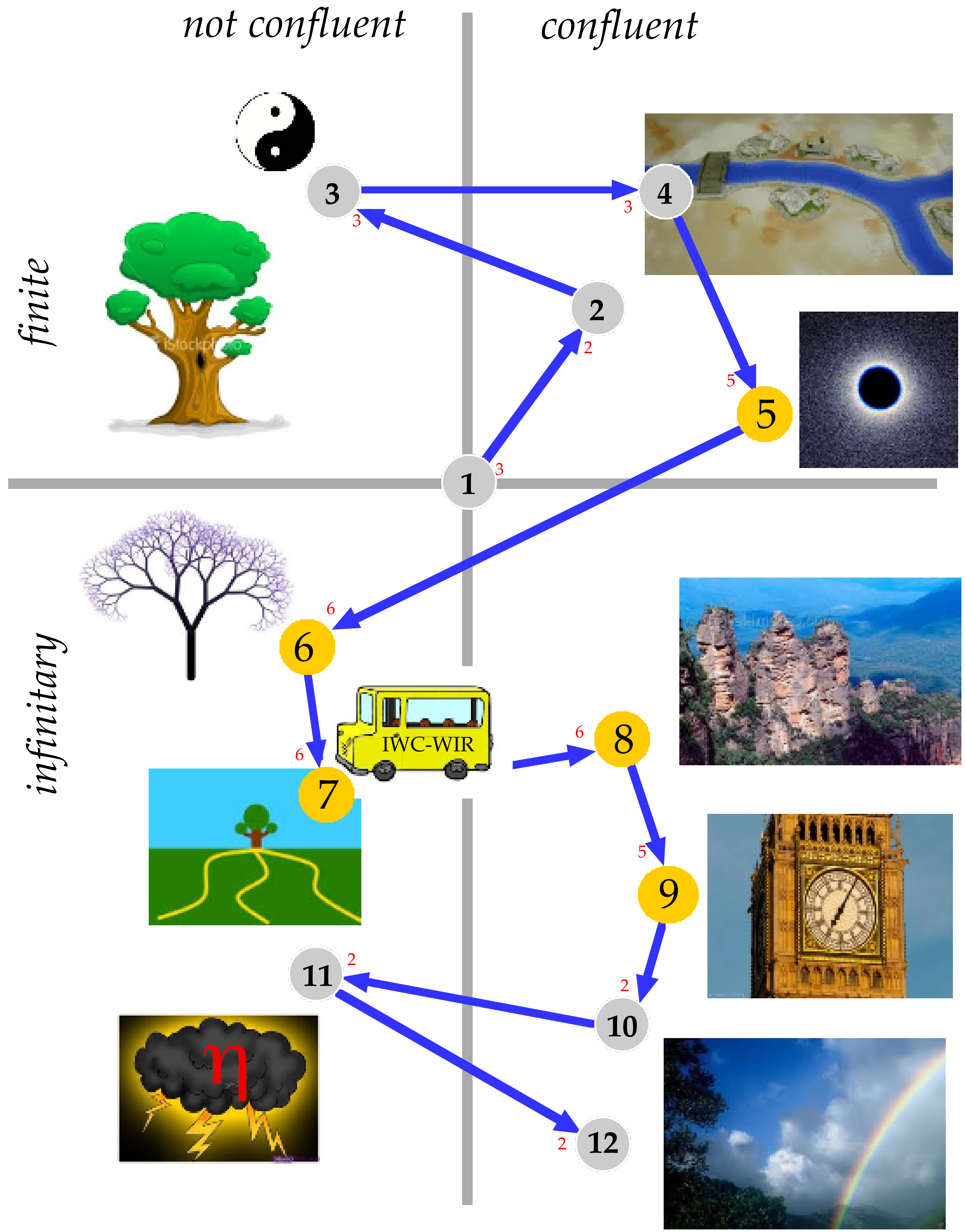


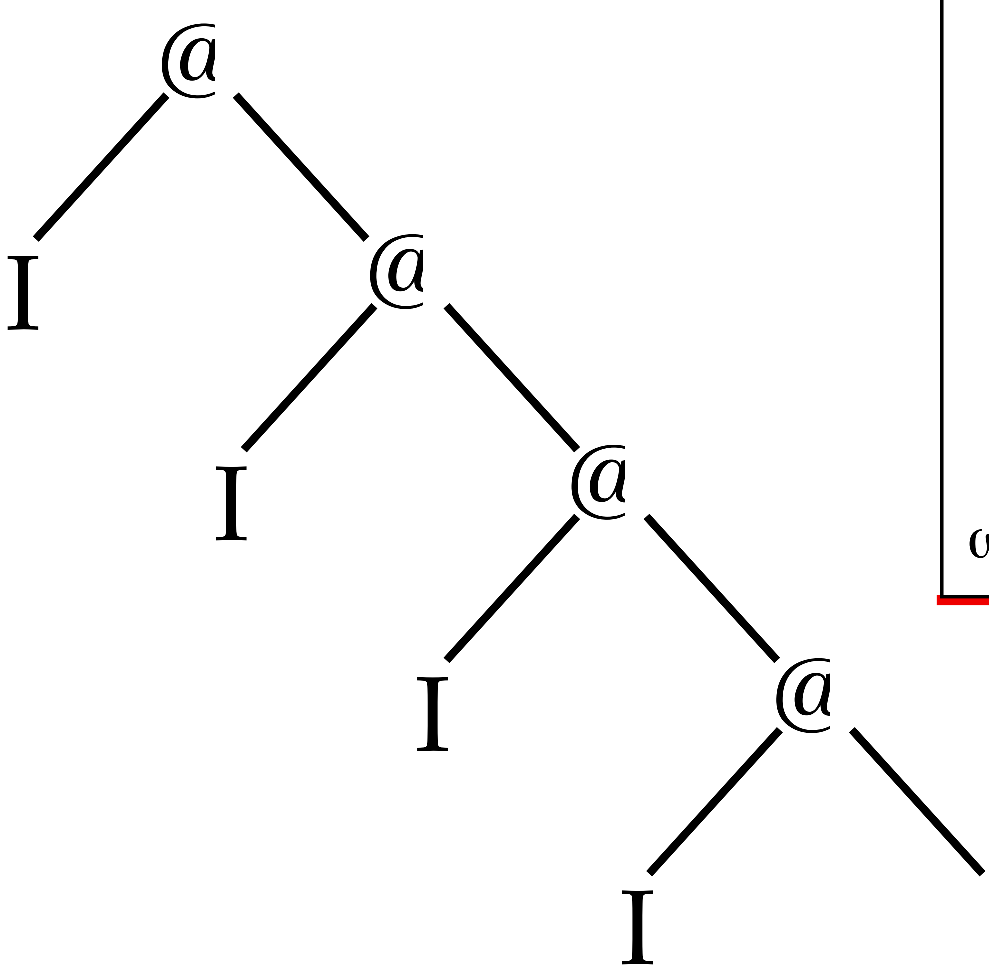
λ^∞ :not PML$^\infty$

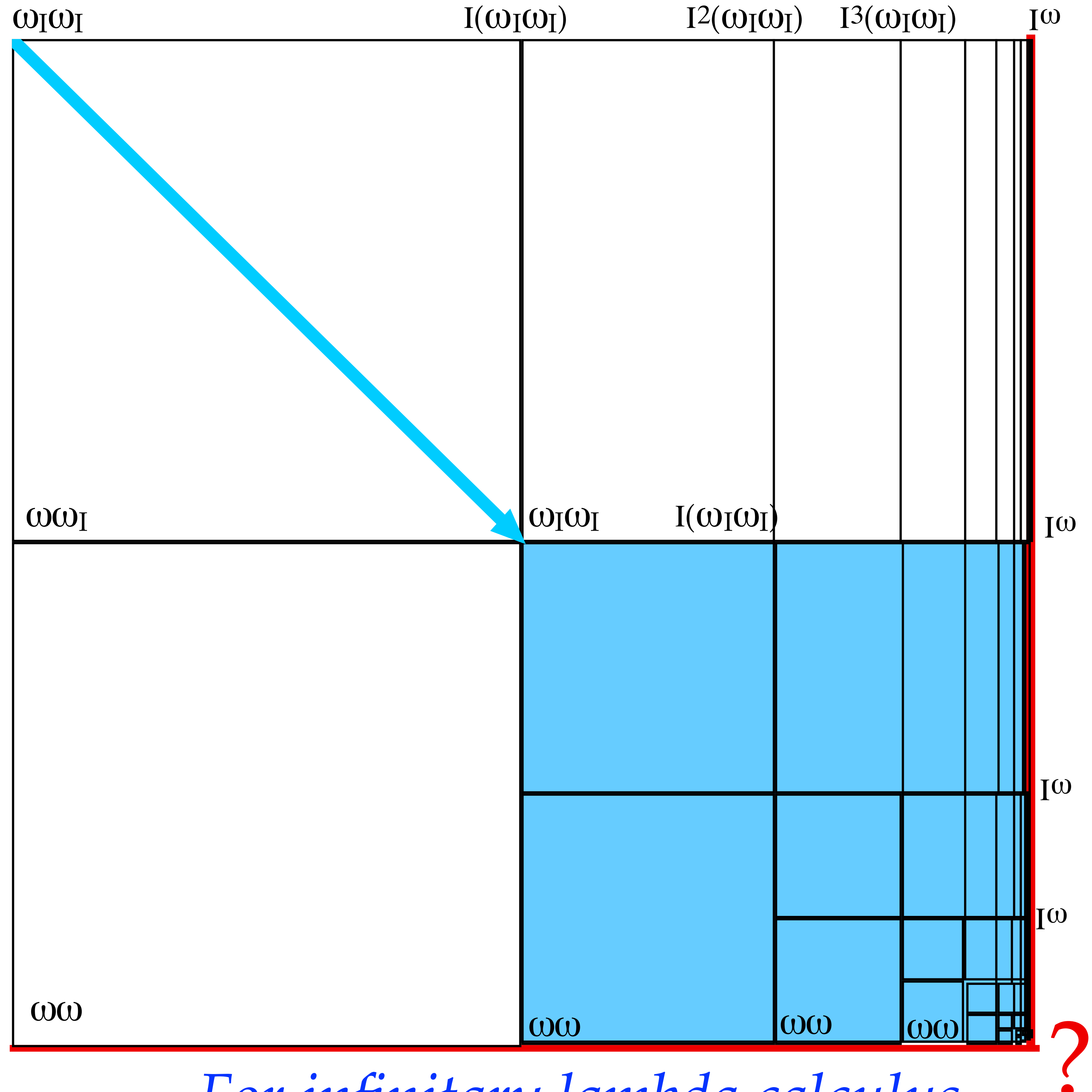
$\omega_I \equiv (\lambda x.I(xx))$

$\omega \equiv \lambda x.xx$

$YI \rightarrow \omega_I\ \omega_I$

$I^\omega \equiv$

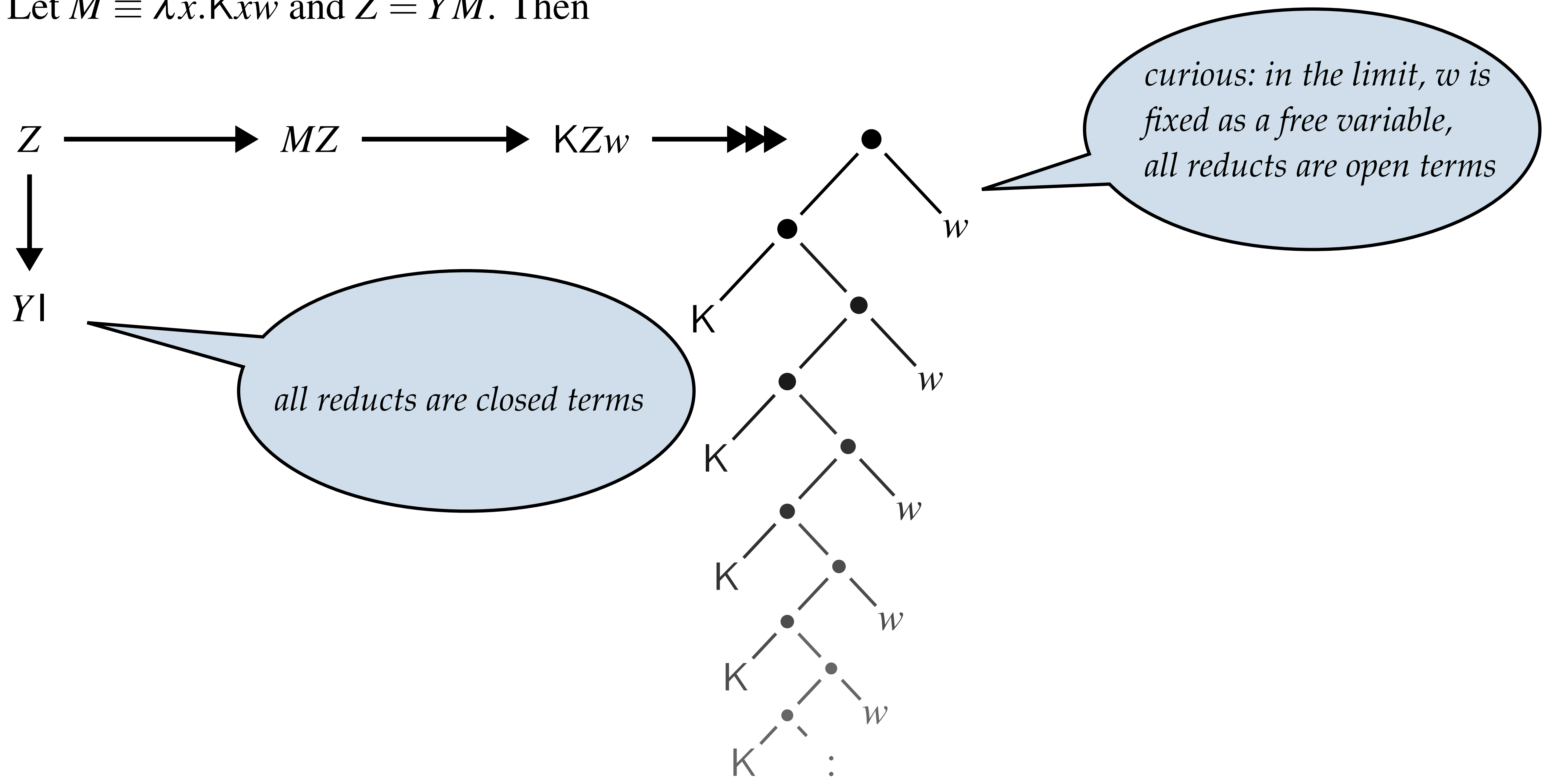




For infinitary lambda calculus Parallel Moves Lemma PML$^\infty$ fails, hence also CR$^\infty$

another counterexample

Let $M \equiv \lambda x.\mathsf{K}xw$ and $Z = YM$. Then



BY $\neq_\beta$? BYS

BYI

BYSI

$\text{BYI} \equiv (\lambda abc.a(bc))\ \text{YI}$

$\text{BYSI} \equiv (\lambda abc.a(bc))\text{YSI}$

$\twoheadrightarrow$

$\twoheadrightarrow$

$\lambda c.\text{Y}(\text{Ic})$

$\rightarrow$

$\lambda c.\text{Yc}$

$\twoheadrightarrow$

Y $\neq_\beta$! Y(SI)

Curry's fpc

Turing's fpc

Y_0: $\lambda f.\ (x.f(xx)(\lambda x.f(xx))$

Y_1: $(\lambda ab.\ b(aab))\ (\lambda ab.\ b(aab))$

$Y_0(SI) \twoheadrightarrow Y_1$

Exercise. Prove that $Y_0 \neq_\beta Y_1$

INFINITARY LAMBDA CALCULUS SUBSUMES SCOTT'S INDUCTION RULE

$$\frac{\Gamma, ax \sqsubseteq bx \vdash a(ux) \sqsubseteq b(ux)}{\Gamma, a\bot \sqsubseteq b\bot \vdash a(\mathsf{Y}u) \sqsubseteq b(\mathsf{Y}u)}$$

$$\mathrm{Yx} \to\to \mathrm{x(Yx)} \to \to \mathrm{x}^2\,\mathrm{(Yx)} \to^{\omega} \mathrm{x}^{\omega} \equiv \mathrm{x(x(x(x\ldots}$$

$$\mathrm{BY} \equiv (\lambda abc.a(bc))\ \mathrm{Y} \qquad =_\infty \qquad \neq_\beta \qquad \mathrm{BYS} \equiv (\lambda abc.a(bc))\ \mathrm{YS}$$

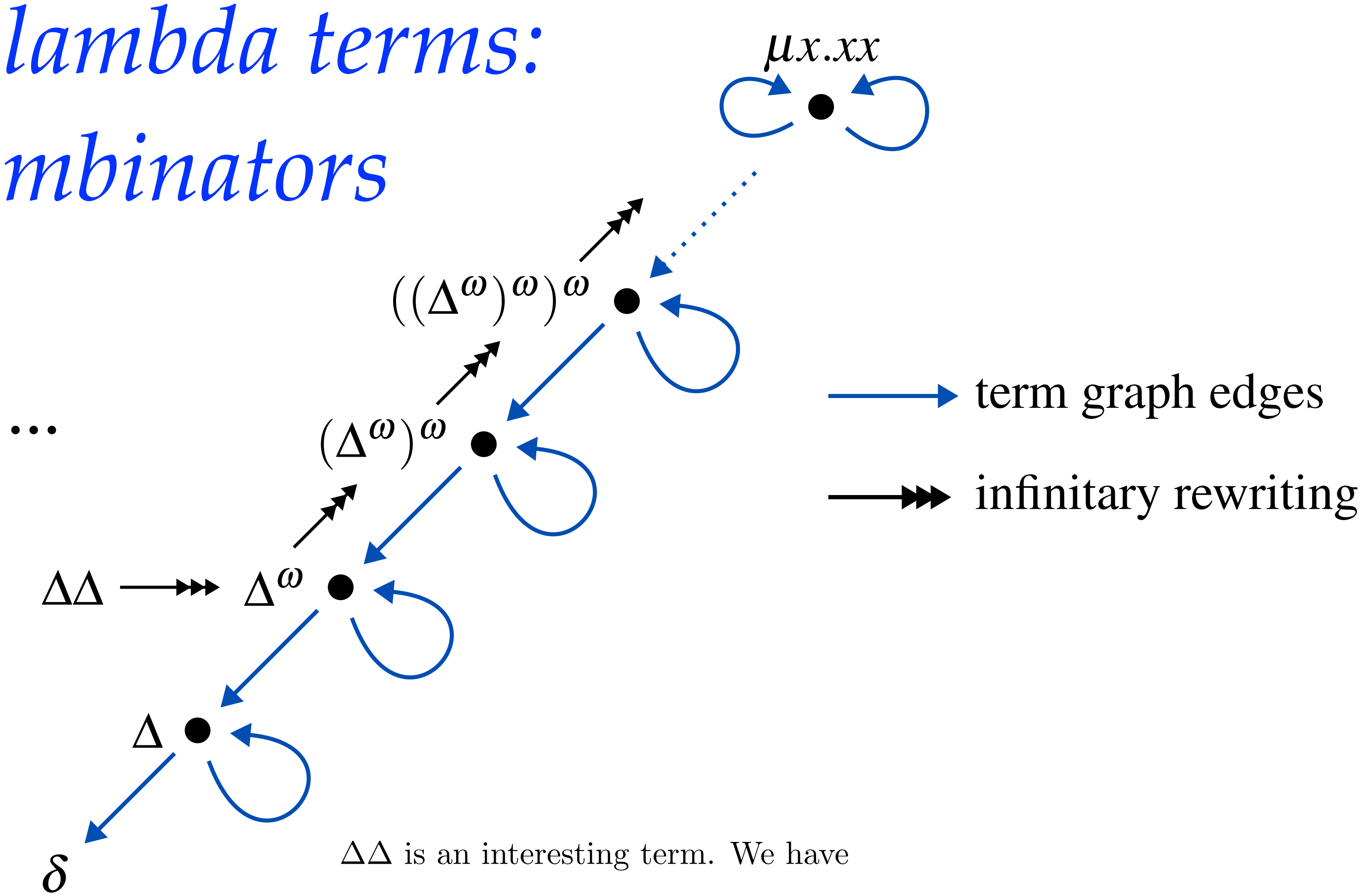
Left: $\mathrm{BY} \to \lambda bc.\mathrm{Y}(bc) \to^{\omega} \lambda bc.\ (bc)^{\omega}$

Right: $\mathrm{BYS} \to\to \lambda c.\mathrm{Y(Sc)} \to\to \lambda c.\ \mathrm{Sc(Y(Sc))} \to\to \lambda cz.\ cz\mathrm{(Y(Sc)}z) \to\to \lambda cz.\ cz(cz\mathrm{(Y(Sc)}z)) \to^{\omega} \lambda cz.\ (cz)^{\omega}$

$$\lambda bc.\ (bc)^{\omega} \equiv \lambda cz.\ (cz)^{\omega}$$

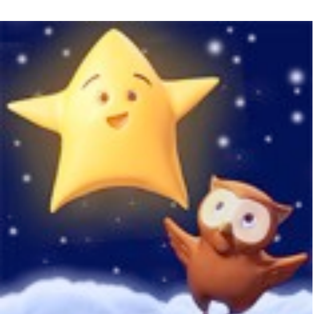
playing with infinite lambda terms: infinite fixed point combinators

twinkle = Δ = δ^ω = δ(δ(δ(δ ...



$$\Delta x \equiv \delta\Delta x \to_\beta \to_\beta x(\Delta x)$$

(SS)^ωSSSI, another infinite fpc



$\Delta\Delta$ is an interesting term. We have

$$\Delta\Delta \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow \Delta^\omega \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow (\Delta^\omega)^\omega \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow ((\Delta^\omega)^\omega)^\omega \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow \cdots$$

See Figure 8. Somewhat surprisingly, $\Delta\Delta$ does have a normal form, viz. $\mu x.xx$; and moreover $\Delta\Delta$ has the property SN^∞. To see that $\mu x.xx$ is indeed the normal form, one may consider the reduction

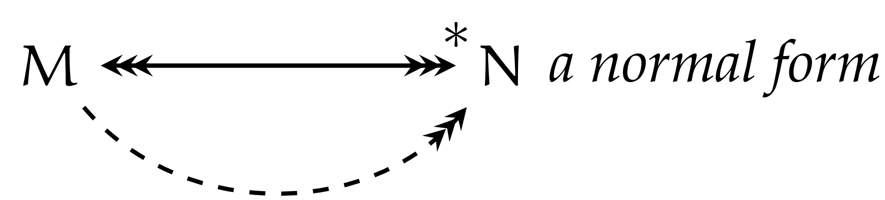
$$\Delta\Delta \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow (\Delta^\omega)^\omega \equiv \Delta^\omega((\Delta^\omega)^\omega) \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow (\Delta^\omega)^\omega((\Delta^\omega)^\omega) \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow \cdots$$

and check that the reductions involved do not employ root redexes. (Only in the reduction $\Delta\Delta \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow \Delta^\omega$ a root step is present; in the 'later' reductions there are no root steps.) In fact we have a strongly convergent reduction

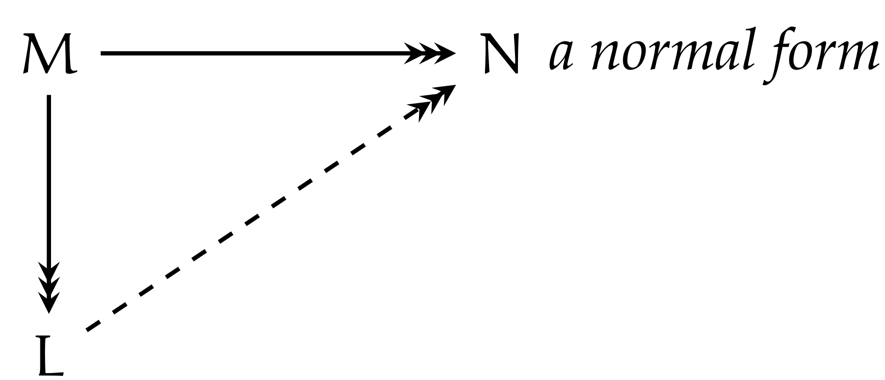
$$\Delta\Delta \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow \Delta^\omega \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow (\Delta^\omega)^\omega \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow ((\Delta^\omega)^\omega)^\omega \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow \cdots \twoheadrightarrow\!\!\!\!\!\twoheadrightarrow \mu x.xx$$

The term $\Delta\Delta$ has uncountably many reducts. It has reductions of any countable ordinal length. It is SN^∞ with $\mu x.xx$ as its unique normal form. This normal form is in fact a Berarducci tree. The example of $\Delta\Delta$ was also mentioned in [4]. SN^∞ can be proved as follows: We have CR^∞ as there are no collapsing rules in this TRS, which is a fragment (sub-TRS) of CL. Since there is a normal form, we have WN^∞. Hence, SN^∞ follows by the equivalence $\mathrm{SN}^\infty \iff \mathrm{WN}^\infty$ as global properties of TRSs.

The infinitary β*-reduction* $\twoheadrightarrow\!\!\!\!\to_\beta$ *has the* infinitary normal form property NF^∞, *that is, for all* $M, N \in \mathit{Ter}^\infty(\lambda)$ *with* N *a normal form and* $M\ (\twoheadleftarrow\!\!\!\!\leftarrow_\beta \cup \twoheadrightarrow\!\!\!\!\to_\beta)^*\ N$ *we have* $M \twoheadrightarrow\!\!\!\!\to_\beta N$. *In a picture:*

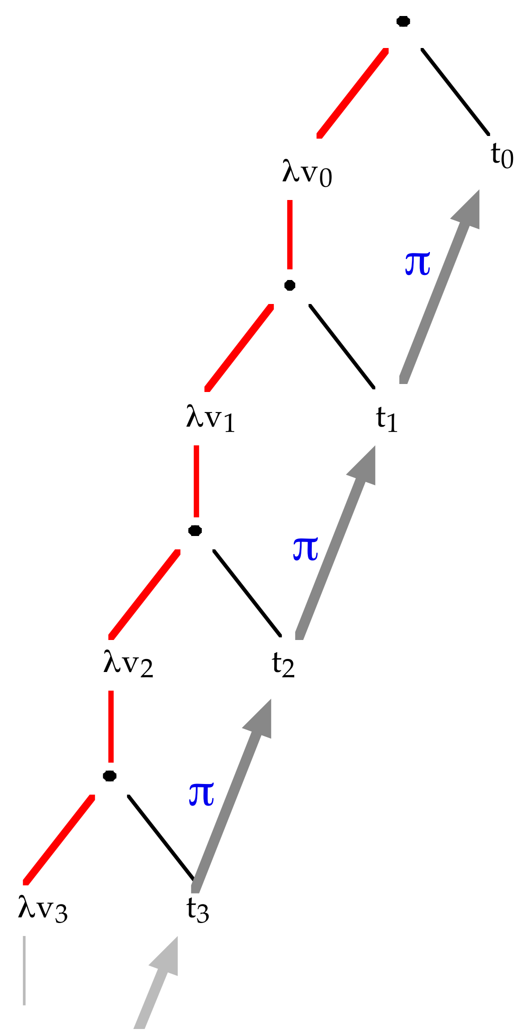


Actually the following property is sufficient:

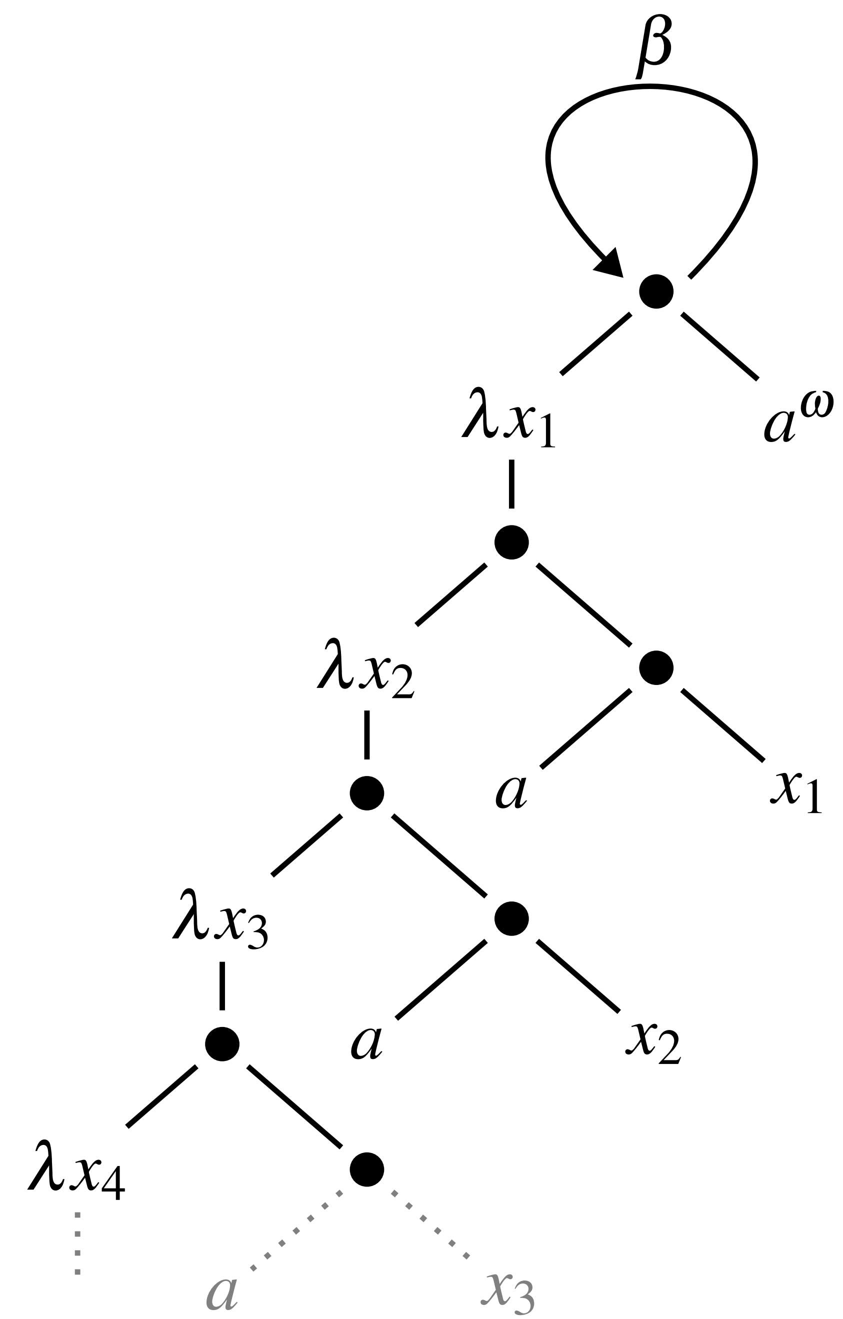


We obtain infinitary unique normal forms UN^∞ as a direct corollary.

playing with infinite lambda terms: looping lambda terms



looping term with infinite spine (red) and cascade projective sequence, with projection π.



Theorem 13.2.6. *In infinitary λ-calculus, a term is root looping if and only if it is of one of the following forms:*

(i) Ω

(ii) I^{ω}

(iii) BB *where* B *is the infinite solution of* $\mathsf{B} = \lambda x.x\mathsf{B}$*,*

(iv) $(\lambda v_0.(\lambda v_1.(\lambda v_2....)t_2)t_1)t_0$ *such that* t_i *is obtained from* t_{i+1} *by replacing* v_0 *by* t_0 *and all variables* v_{j+1} *by* v_j*. We call such a term a* cascade*.*

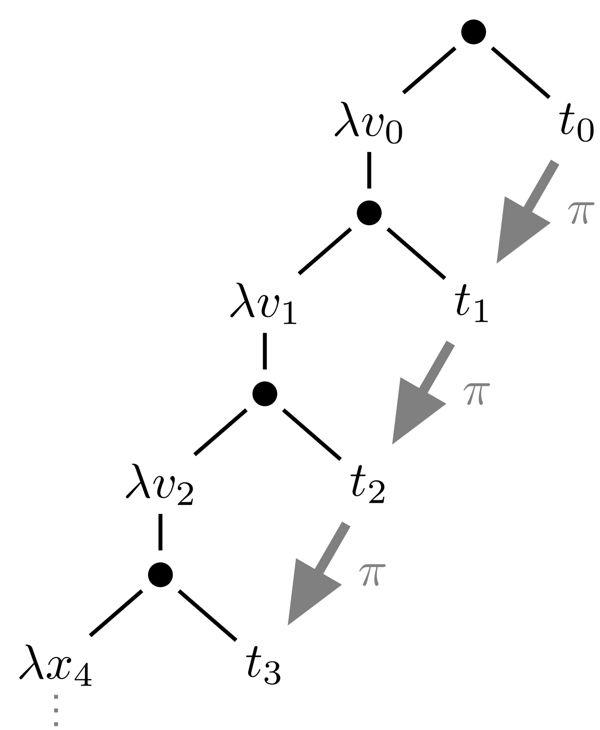
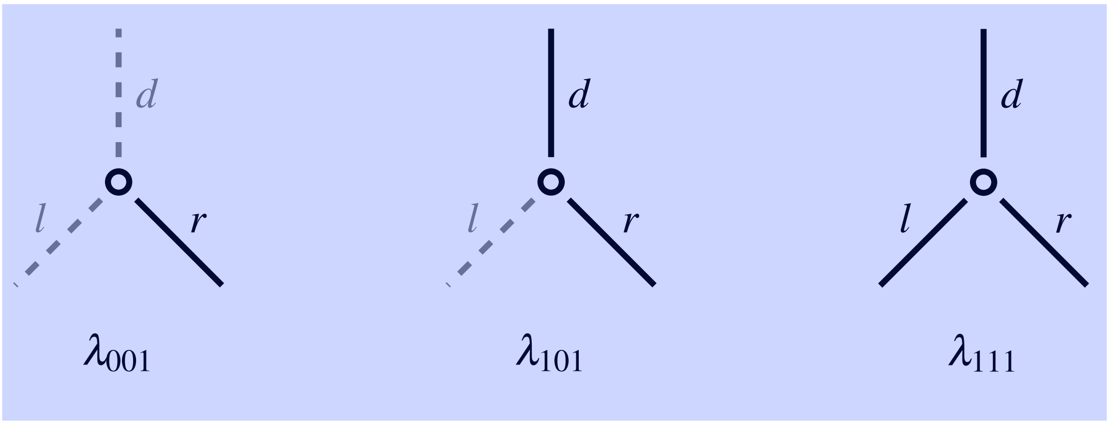
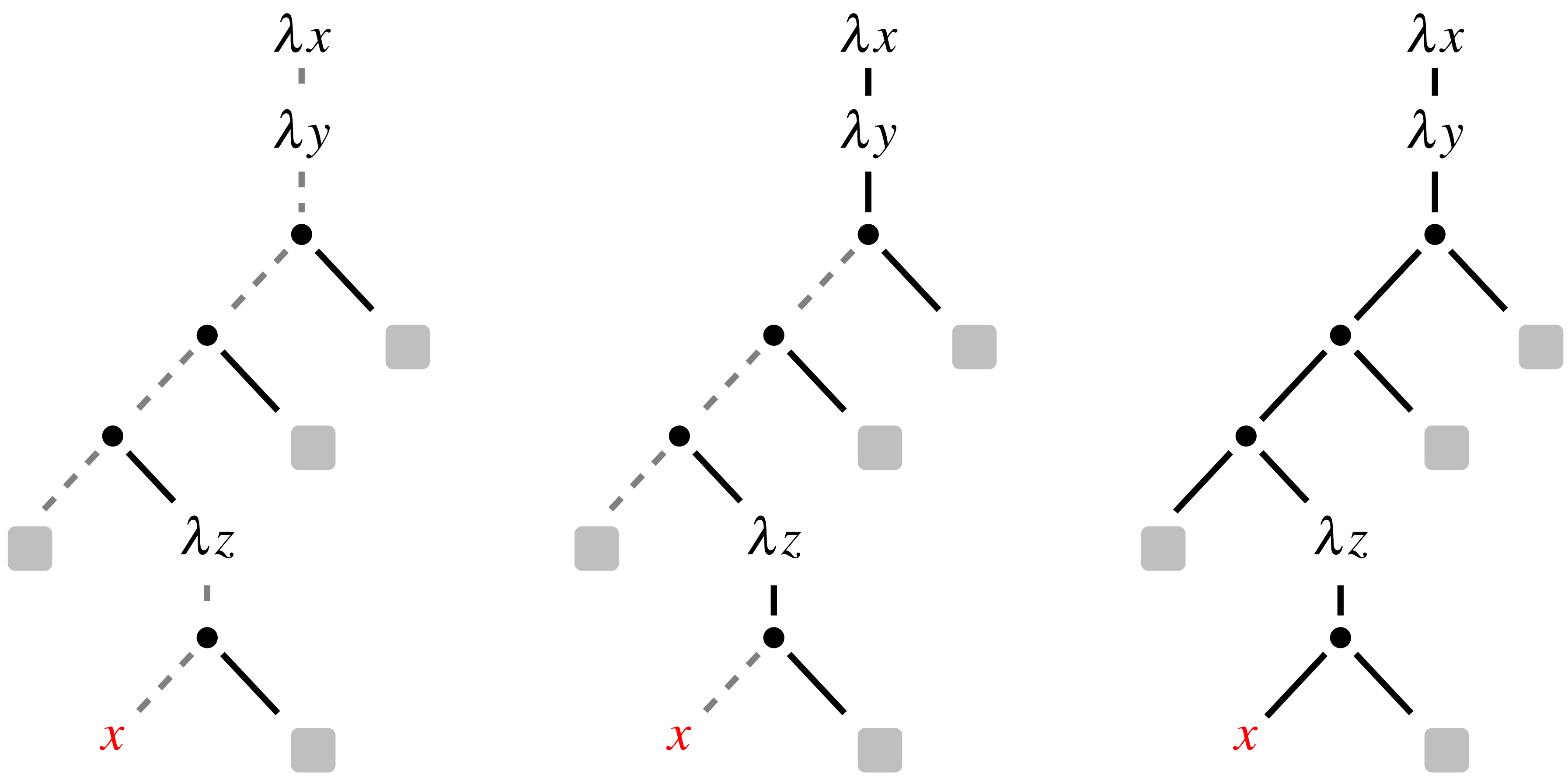


Figure 13.2: *The shape of cascades; here* π *stands for replacing all variables* v_j *by* v_{j+1} *followed by replacing an arbitrary (possibly infinite) number of occurrences of* t_0 *by* v_0*.*

different ways to count depth



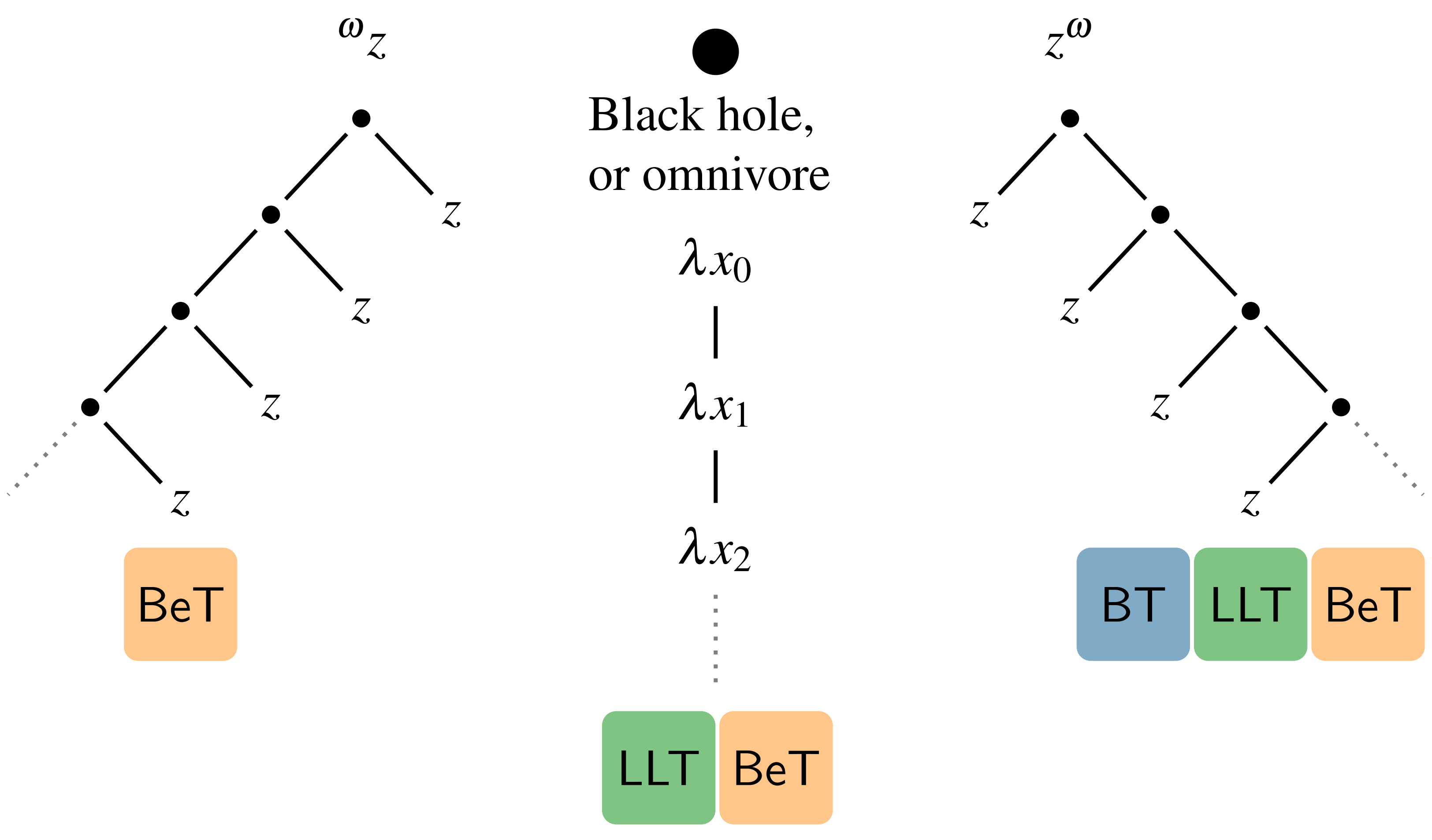


001-*depth* 1
$\{l, d\}$-*steps don't count*

101-*depth* 4
$\{l\}$-*steps don't count*

111-*depth* 7
all steps count

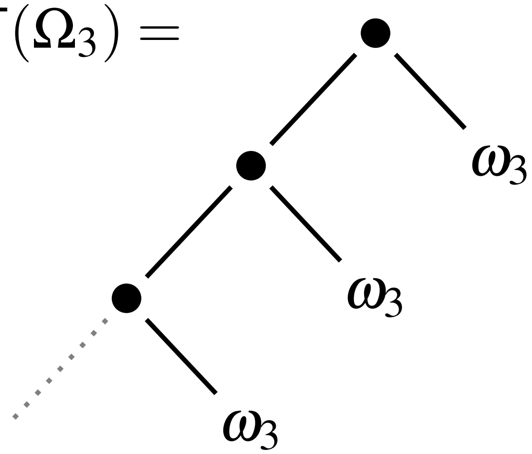
typical terms in the three domains



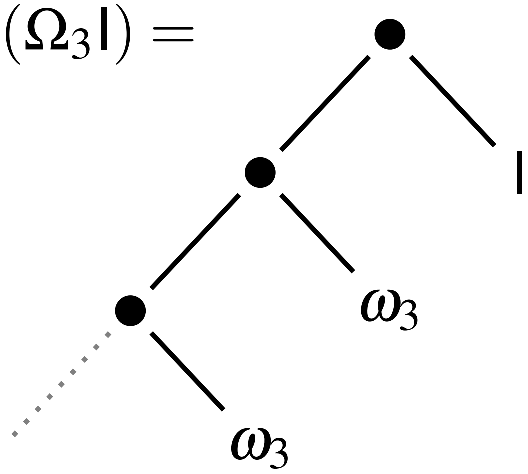
a tool for consistency analysis

Berarducci Trees

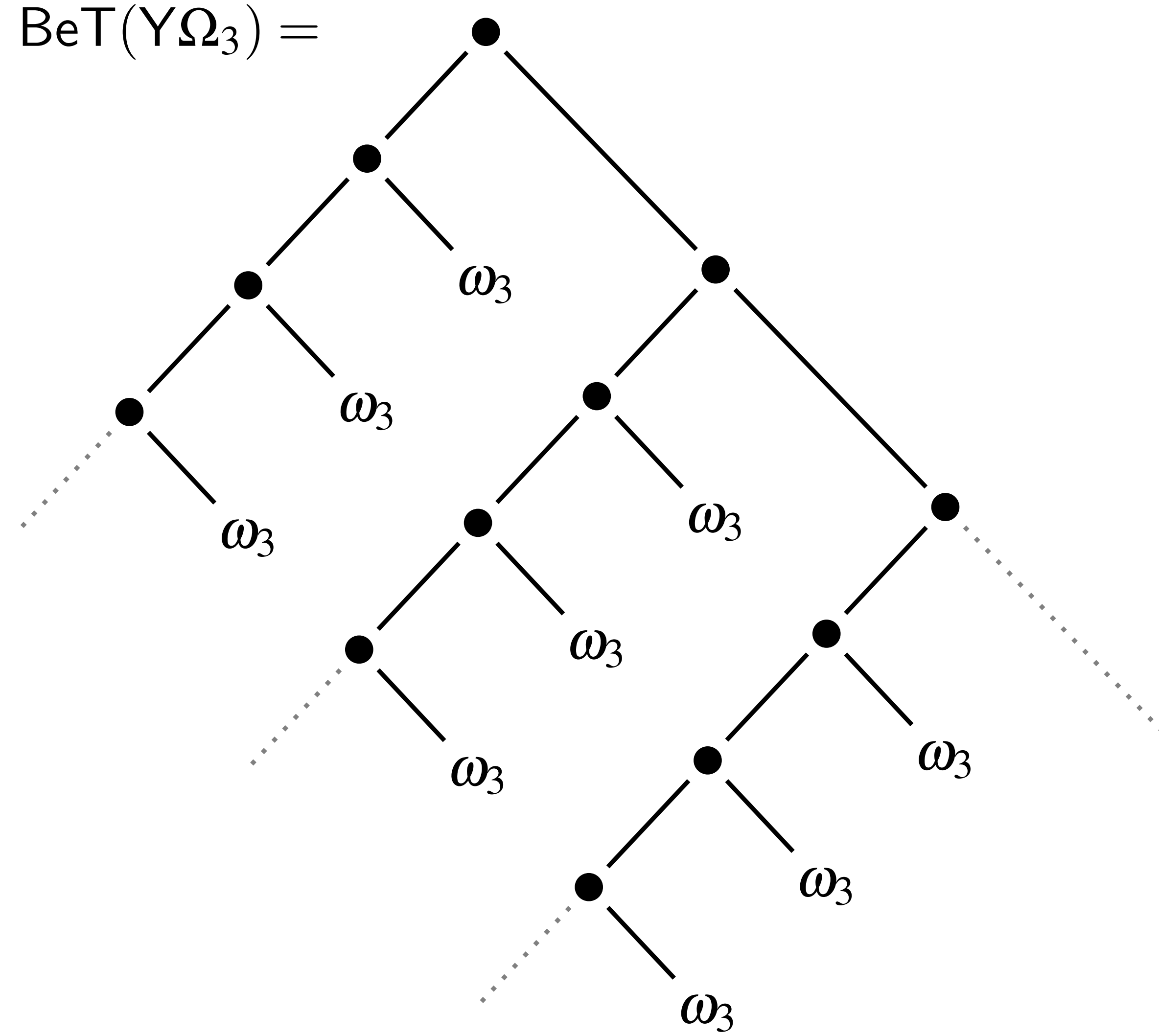
$\mathsf{BeT}(\Omega_3) =$



$\mathsf{BeT}(\Omega_3 \mathsf{I}) =$



$\mathsf{BeT}(\mathsf{Y}\Omega_3) =$

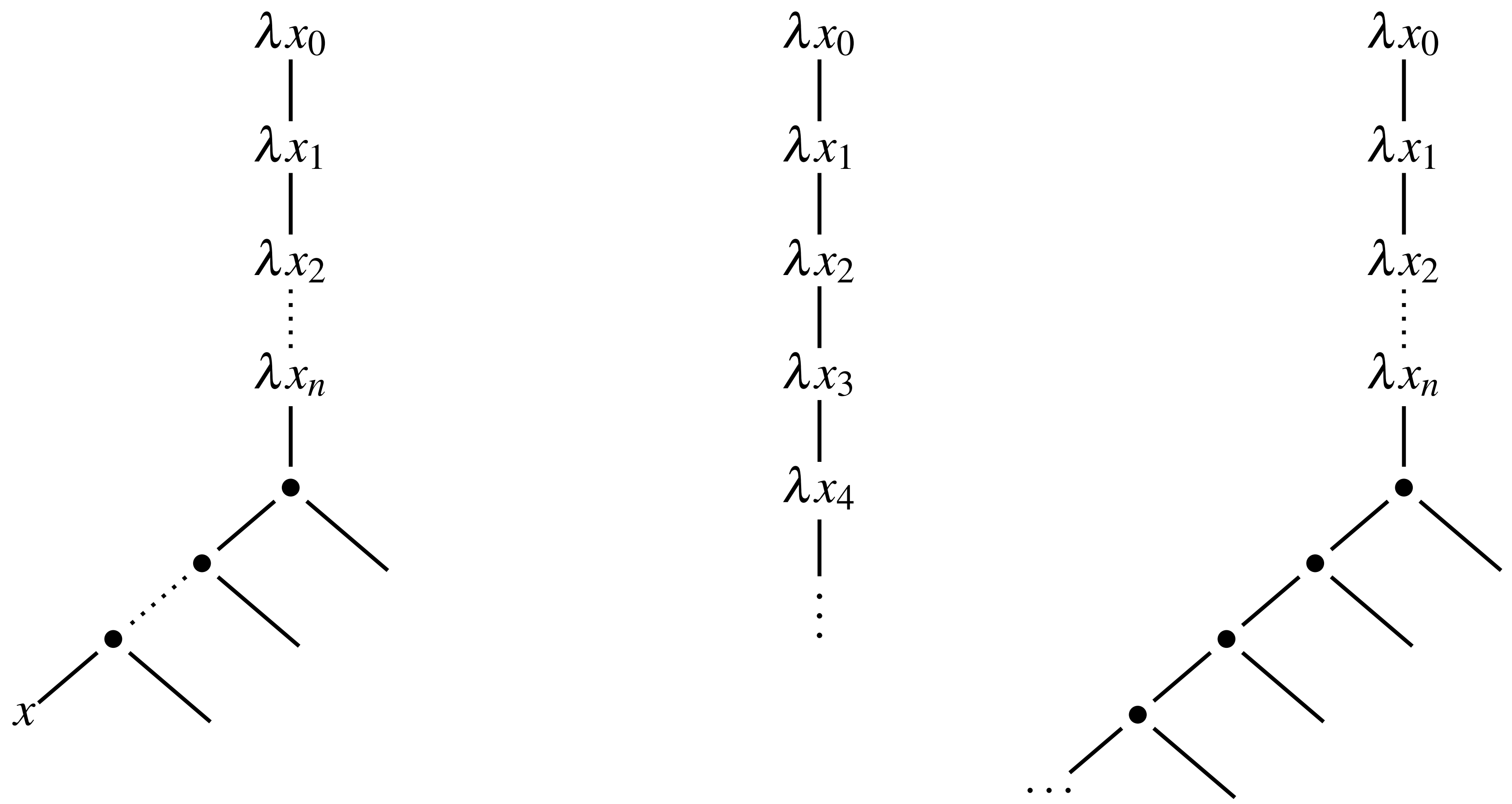


not easy

easy

easy for closed normal forms;
open problem for general terms

building blocks for infinitary lambda normal forms



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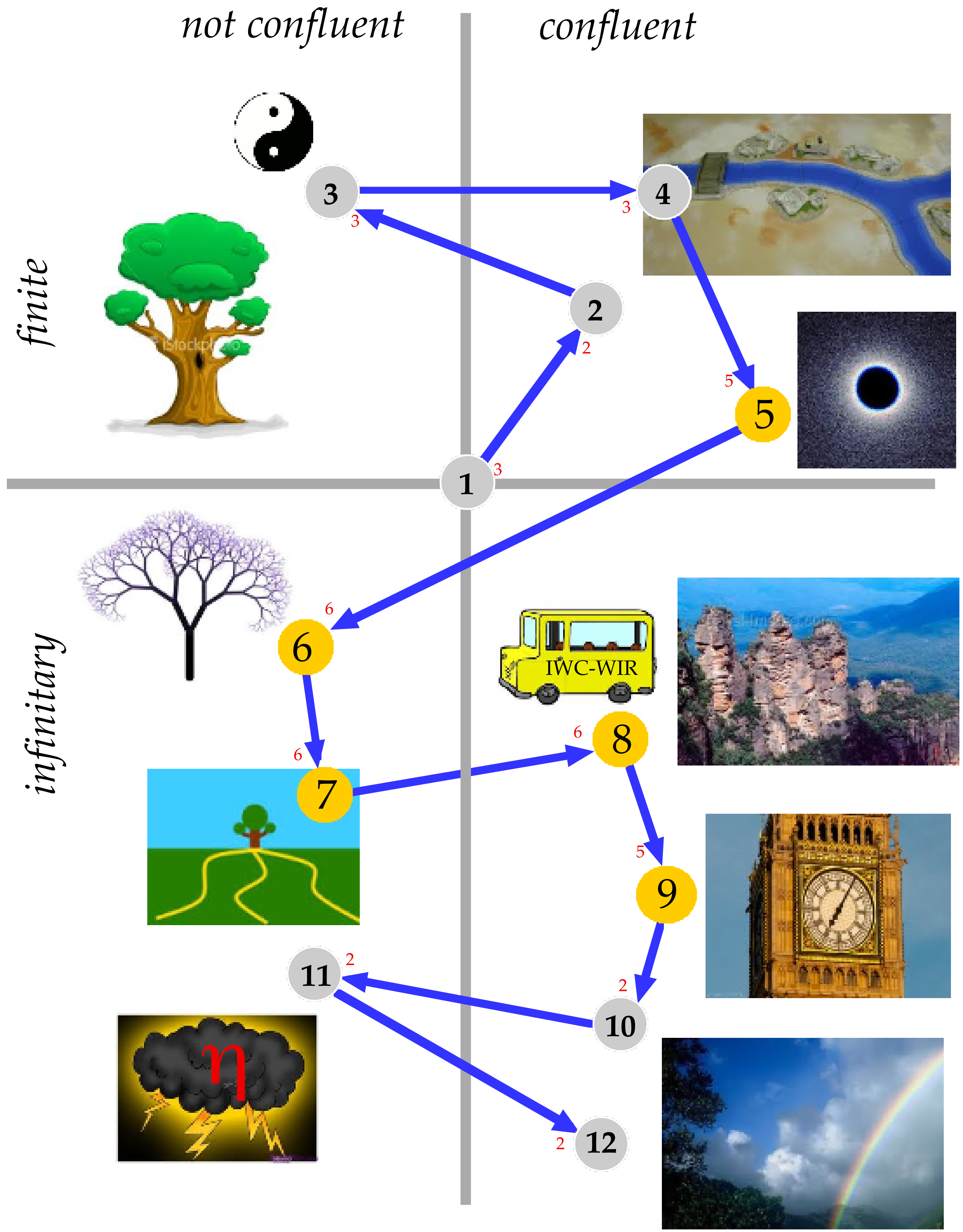


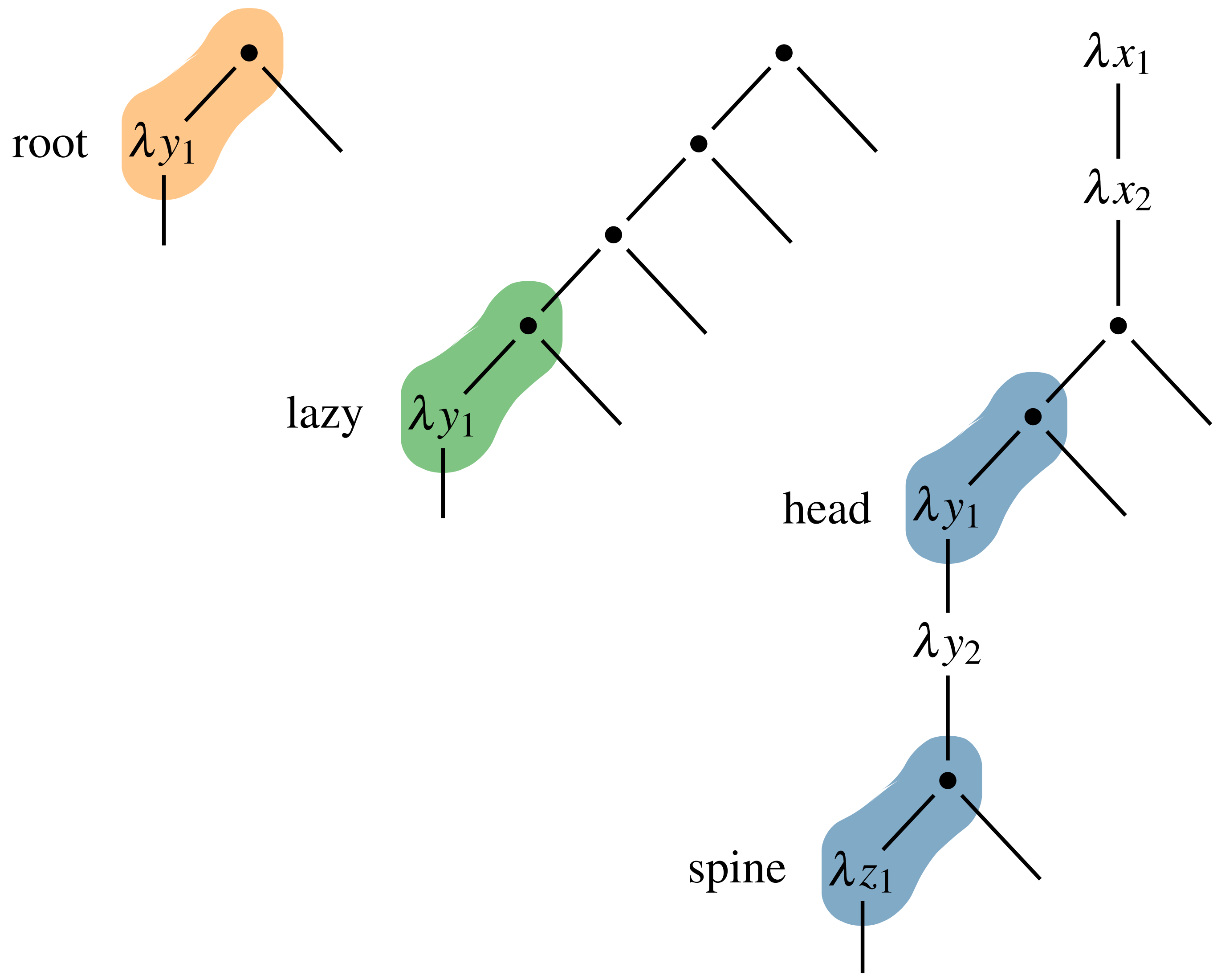
Coinductive definition of BT, LLT, BeT

$$\mathsf{BT}(M) = \begin{cases} \lambda\vec{x}.y\,\mathsf{BT}(M_1)\ldots\mathsf{BT}(M_m) & \text{if } M \text{ has hnf } \lambda\vec{x}.yM_1\ldots M_m, \\ \bot & \text{otherwise.} \end{cases}$$

$$\mathsf{LLT}(M) = \begin{cases} x\,\mathsf{LLT}(M_1)\ldots\mathsf{LLT}(M_m) & \text{if } M \text{ has whnf } x\,M_1\ldots M_m, \\ \lambda x.\mathsf{LLT}(M') & \text{if } M \text{ has whnf } \lambda x.M', \\ \bot & \text{otherwise.} \end{cases}$$

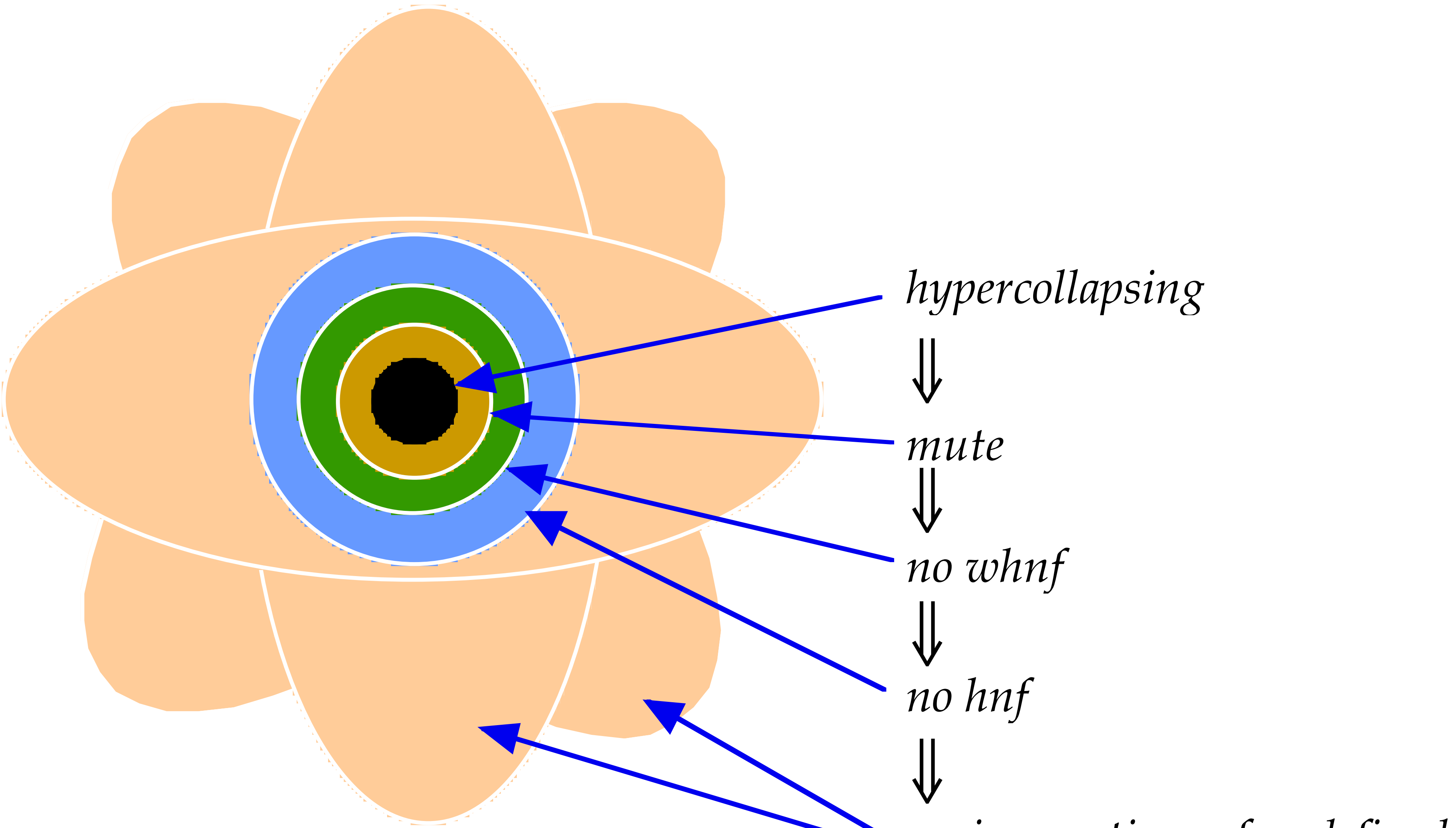
$$\mathsf{BeT}(M) = \begin{cases} y & \text{if } M \twoheadrightarrow y, \\ \lambda x.\mathsf{BeT}(N) & \text{if } M \twoheadrightarrow \lambda x.N, \\ \mathsf{BeT}(M_1)\,\mathsf{BeT}(M_2) & \text{if } M \twoheadrightarrow M_1\,M_2 \text{ such that } M_1 \text{ is of order } 0, \\ \bot & \text{in all other cases (i.e., when } M \text{ is mute).} \end{cases}$$

the typical redexes

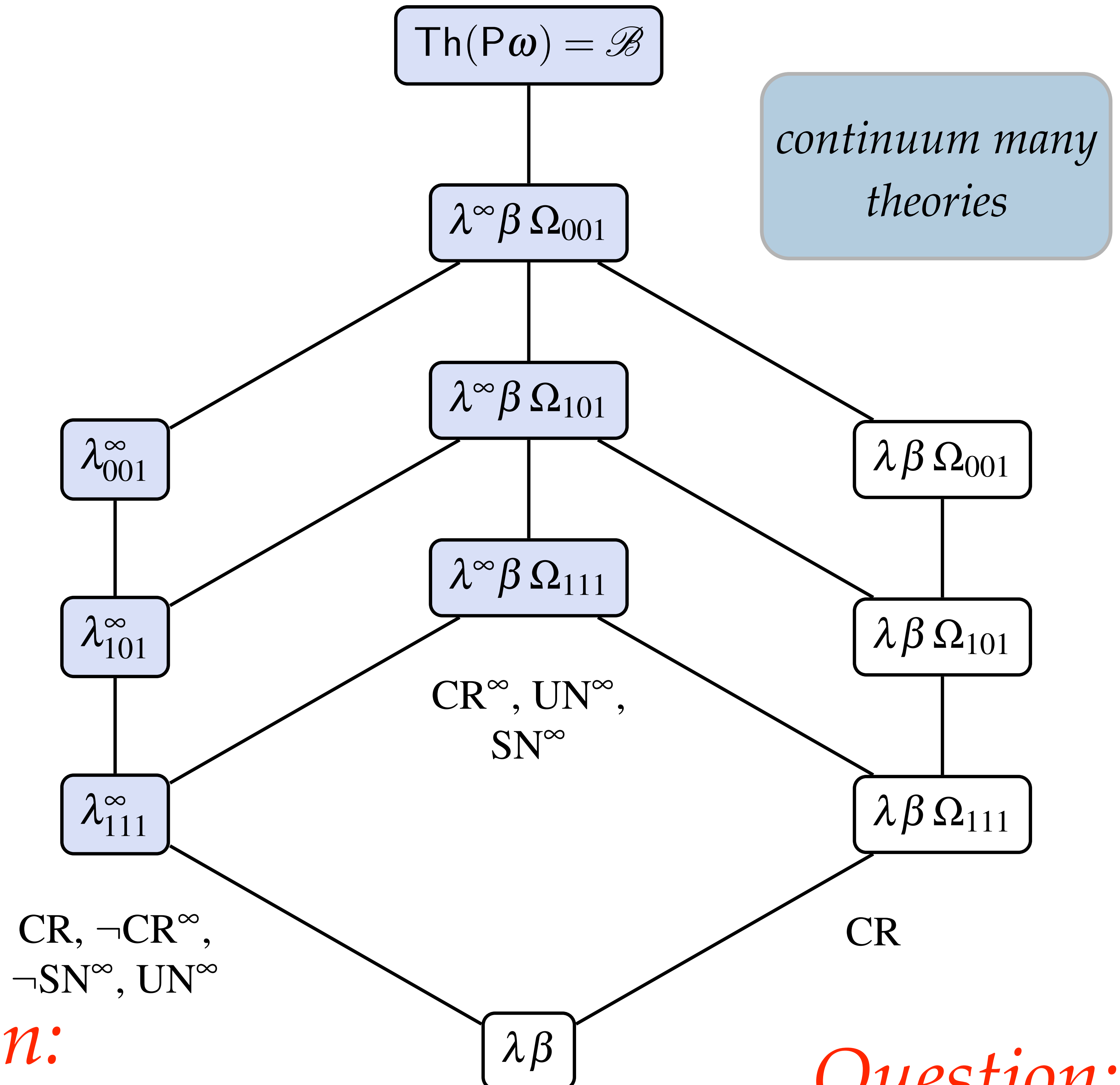


redex is root $\Rightarrow$ lazy $\Rightarrow$ head $\Rightarrow$ spine

notions of undefinedness, with a caveat



lambda theories compared



Question:

Do we have LLT and BeT versions of $\mathsf{P}\omega$?

Question:

can we interpret $\lambda^\infty\beta\Omega$ *in* $\mathsf{P}\omega$?

1. *Introduction*
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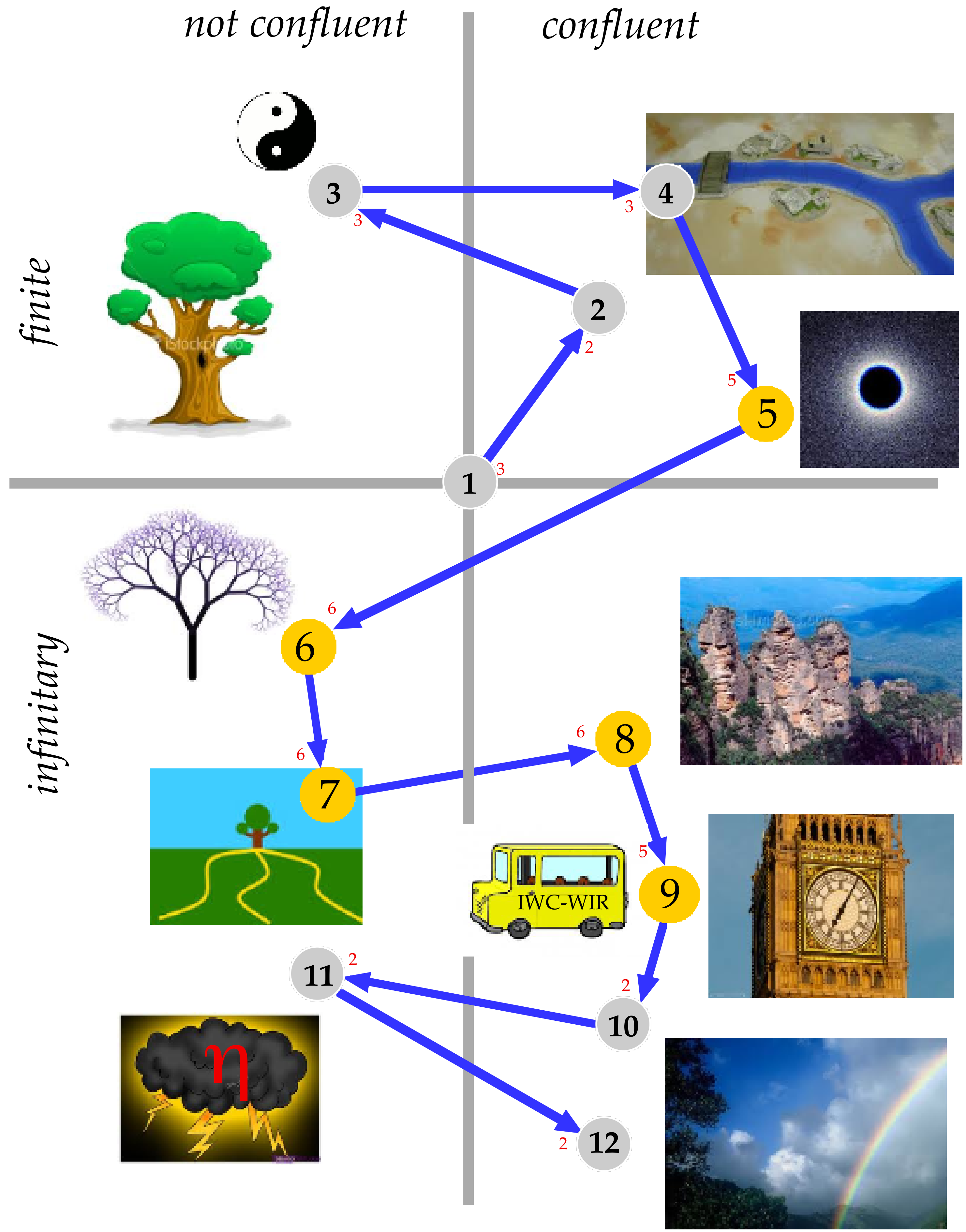
4. *Confluence of a higher order*
5. *Lambda with black holes: confluence*
6. *Confluence lost in infinity*
7. *The threefold path*
8. *Black holes to the rescue*
9. *The rhythm of lambda terms*
10. *Getting rid of ordinals*
11. *Infinity and eta: total breakdown*
12. *A lambda universe*



clock behaviour of fpc in Böhm sequence of fpc's

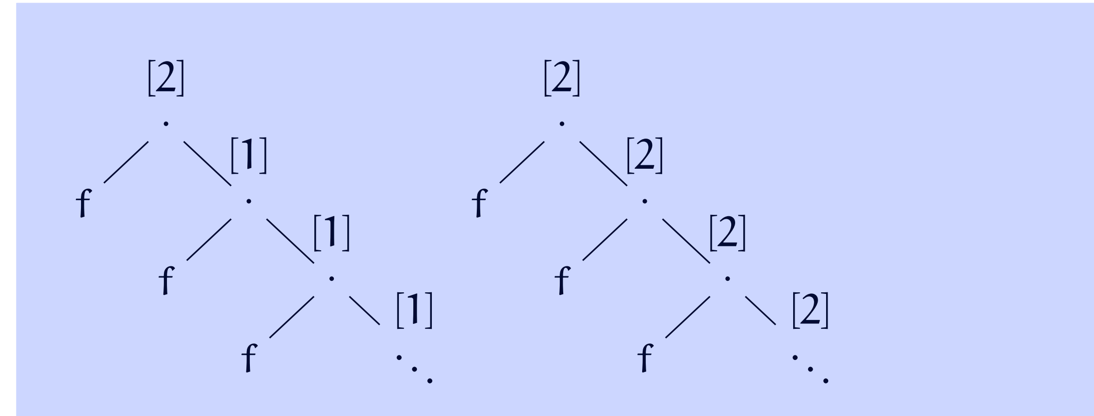
Y_0, $\mathsf{Y}_0\,\delta$, $\mathsf{Y}_0\,\delta\delta$, $\mathsf{Y}_0\,\delta\delta\delta$, $\mathsf{Y}_0\,\delta\delta\delta\delta$, ...

$$\mathsf{Y}_3 \equiv \mathsf{Y}_0 \delta\delta\delta \xrightarrow[h]{7} \lambda a.a(\omega_\delta \omega_\delta \delta\delta a)$$

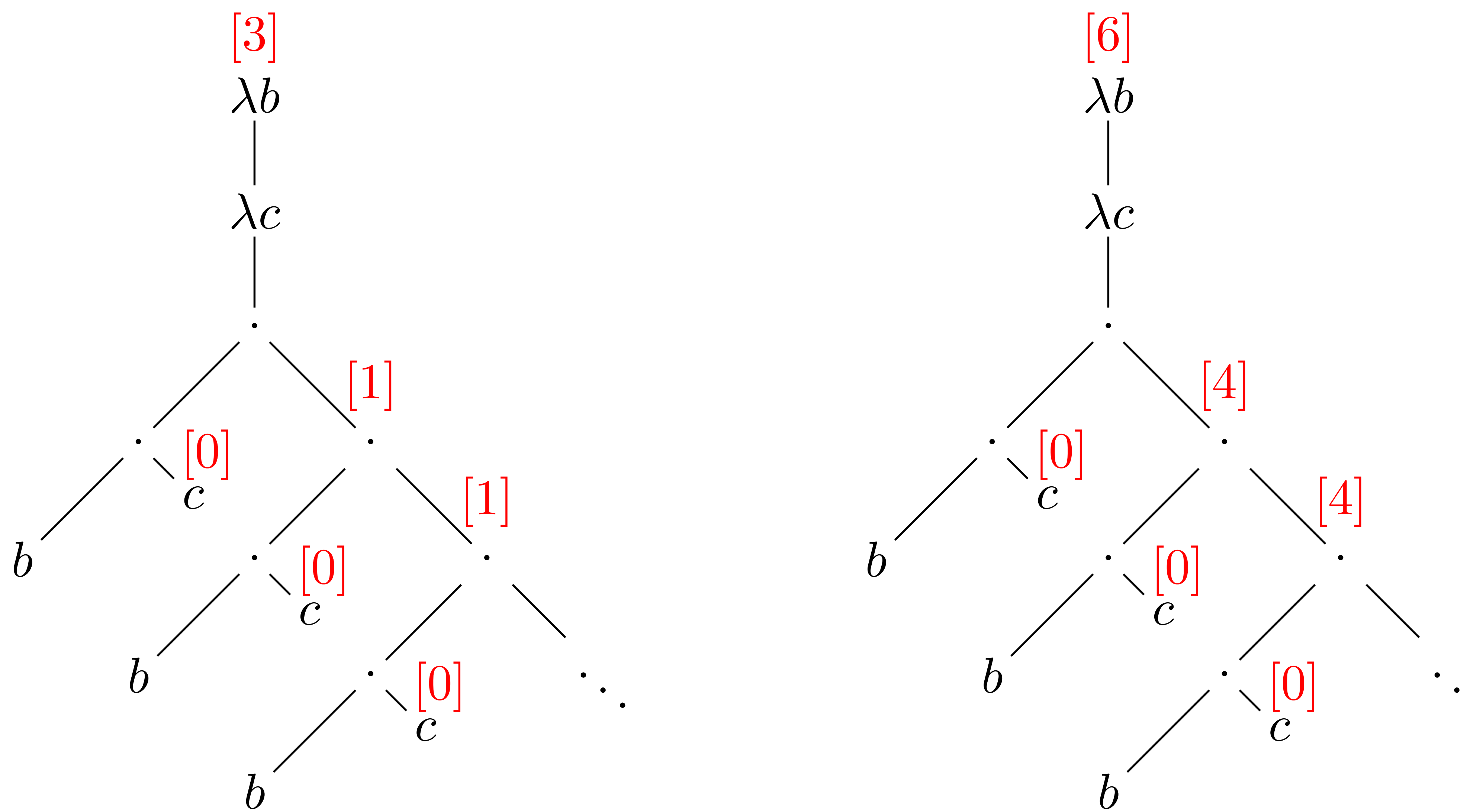
$\lambda a.a\Box$

$$\omega_\delta \omega_\delta \delta\delta a \xrightarrow[h]{7} a(\omega_\delta \omega_\delta \delta\delta a)$$

$a\Box$



Clocked BT's of $Y_0 f$ *and* $Y_1 f$



Clocked Böhm trees of BY_0 *and* $\mathsf{BY}_0\mathsf{S}$.

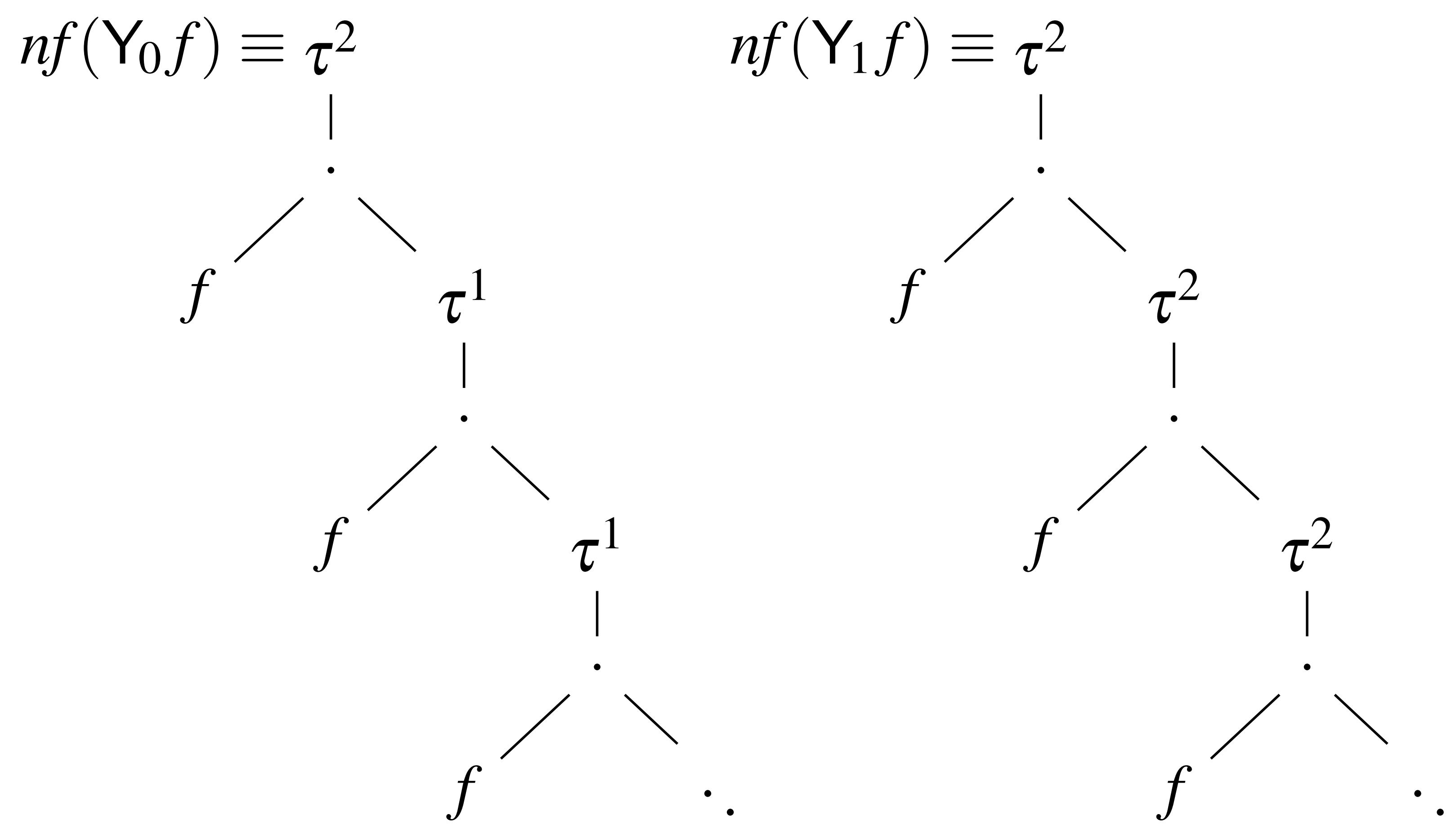
Clocked Lambda Calculus

$$(\lambda x.M)N \to \tau(M[x:=N])$$
$$\tau(M)N \to \tau(MN)$$

The τ's are ticks of the clock (measure of efficiency).

Properties: orthogonal, SN^∞, CR^∞, UN^∞

Normal forms are clocked Lévy–Longo trees:



different clock

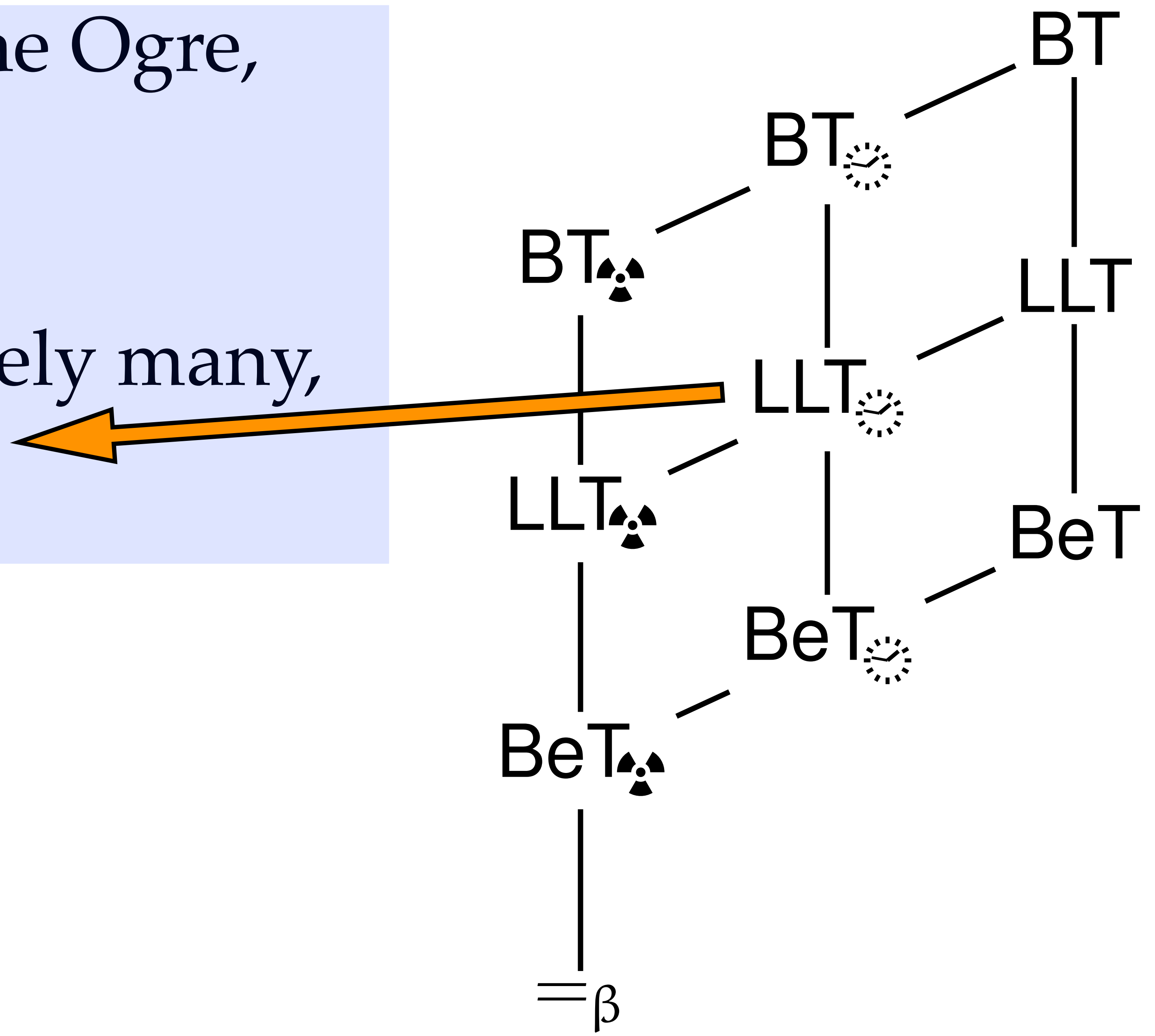
$\Rightarrow \; \mathsf{Y}_0 \neq \mathsf{Y}_1$

clocked lambda theories

Exercise.

(i) in $\lambda^\infty\beta$ there is only one Ogre, Omnivore;

(ii) in $\lambda\beta$ there are infinitely many, i.p. all Y_nK are different



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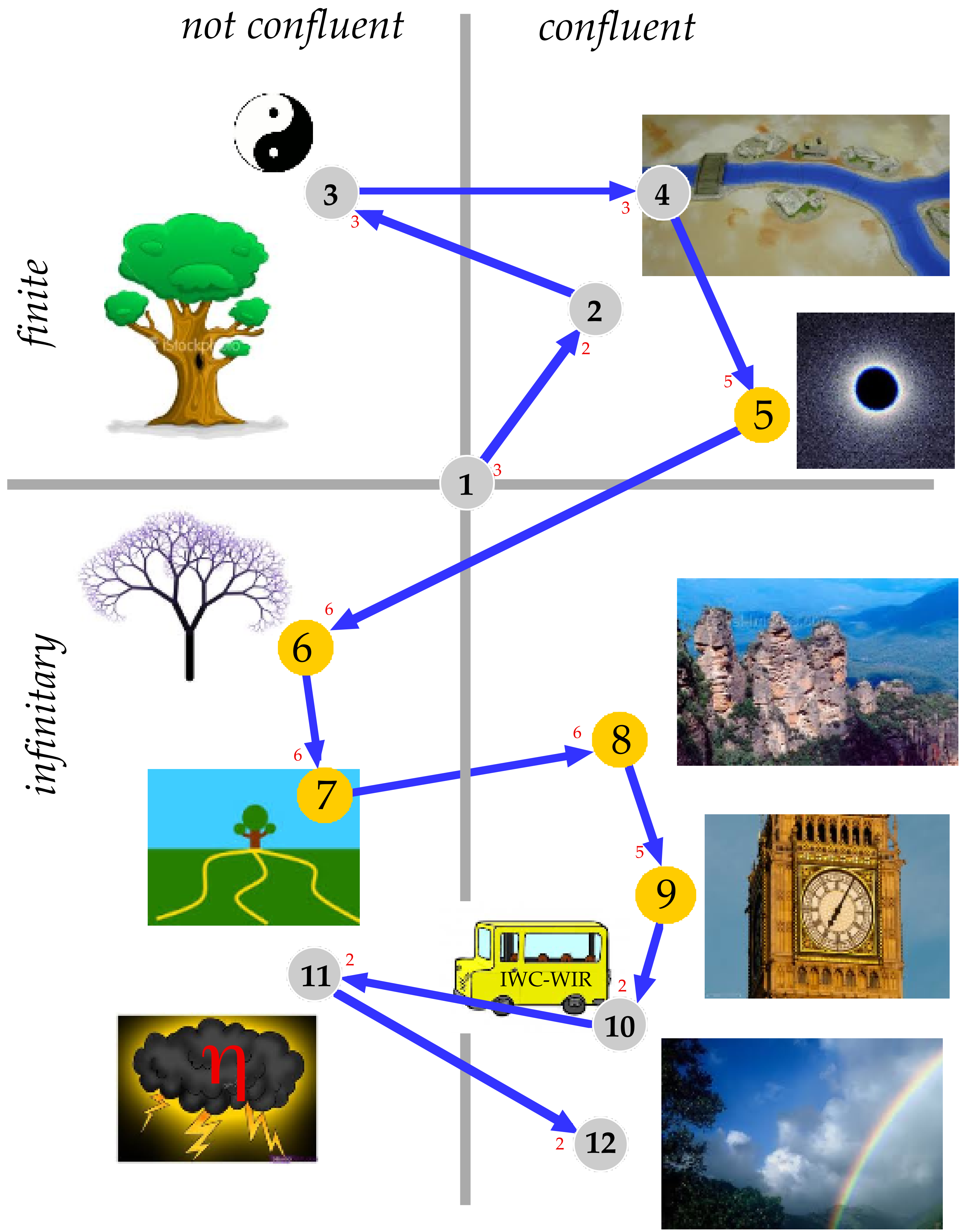

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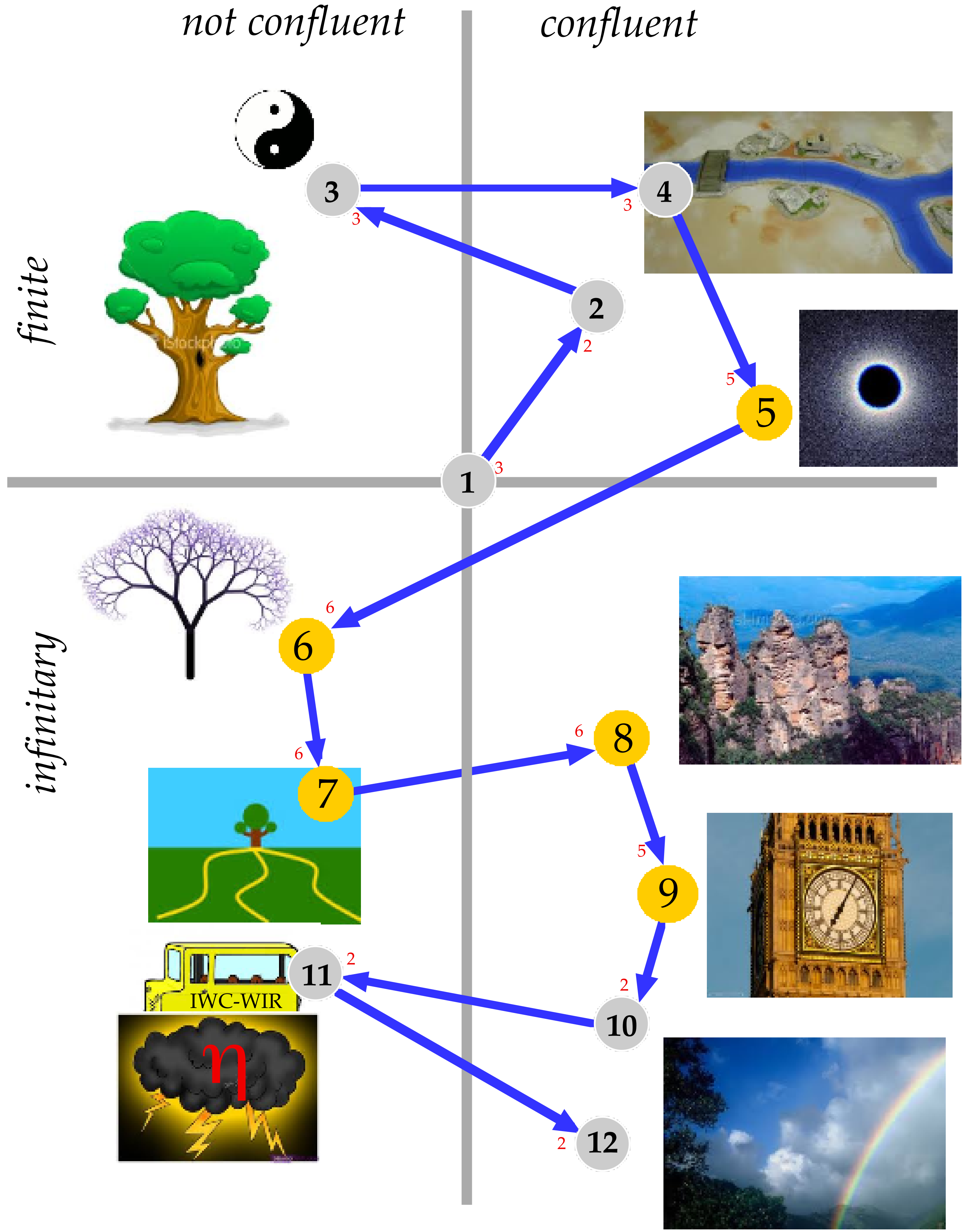


$$\mathsf{PS} \to \varepsilon \qquad\qquad \mathsf{SP} \to \varepsilon$$

where ε is the empty word. This system has two trivial critical pairs:

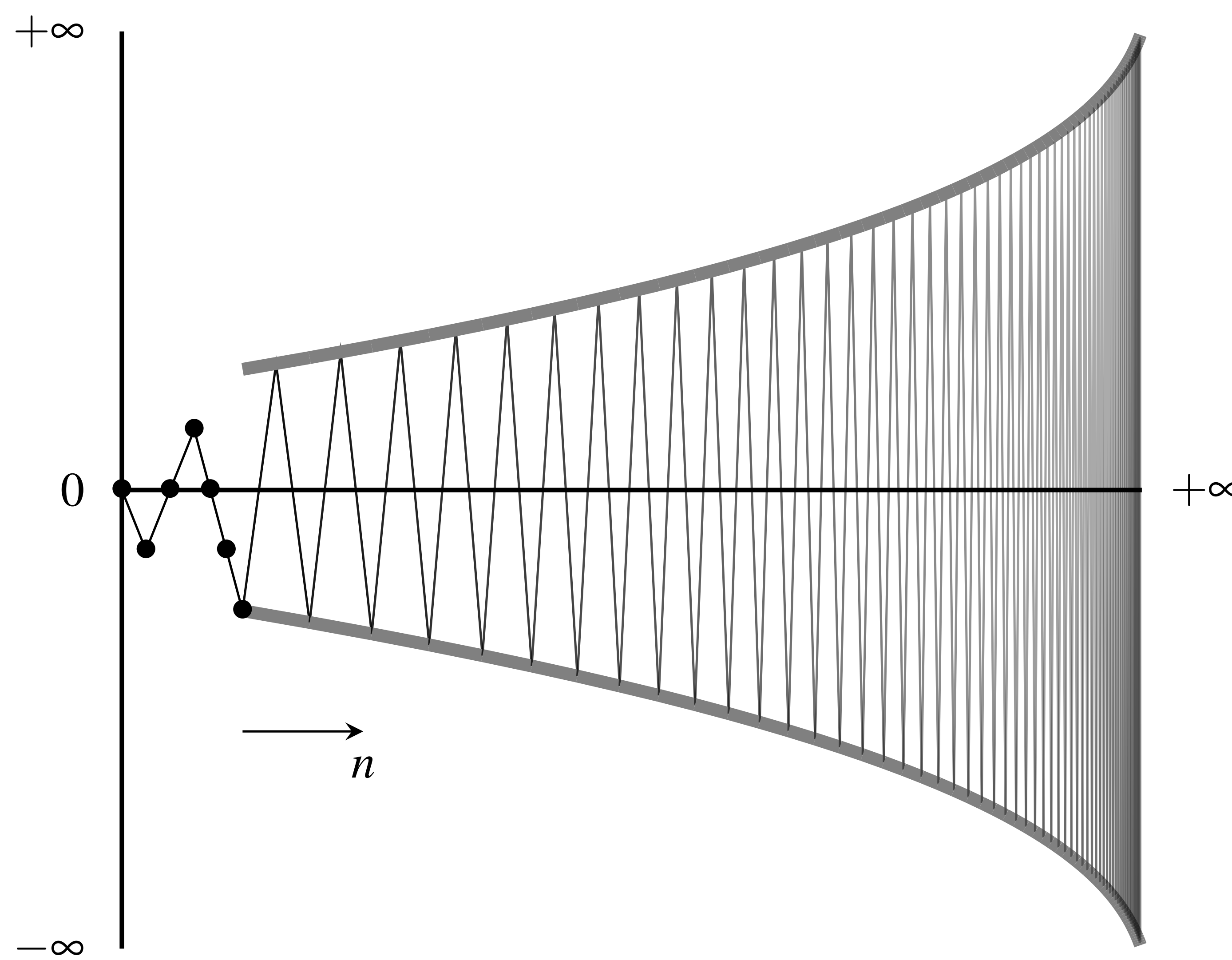
$$\mathsf{P} \leftarrow \underline{\mathsf{P}\overline{\mathsf{S}}}\overline{\mathsf{P}} \to \mathsf{P} \qquad\qquad \mathsf{S} \leftarrow \underline{\mathsf{S}\overline{\mathsf{P}}}\overline{\mathsf{S}} \to \mathsf{S}\,,$$

and hence is weakly orthogonal.

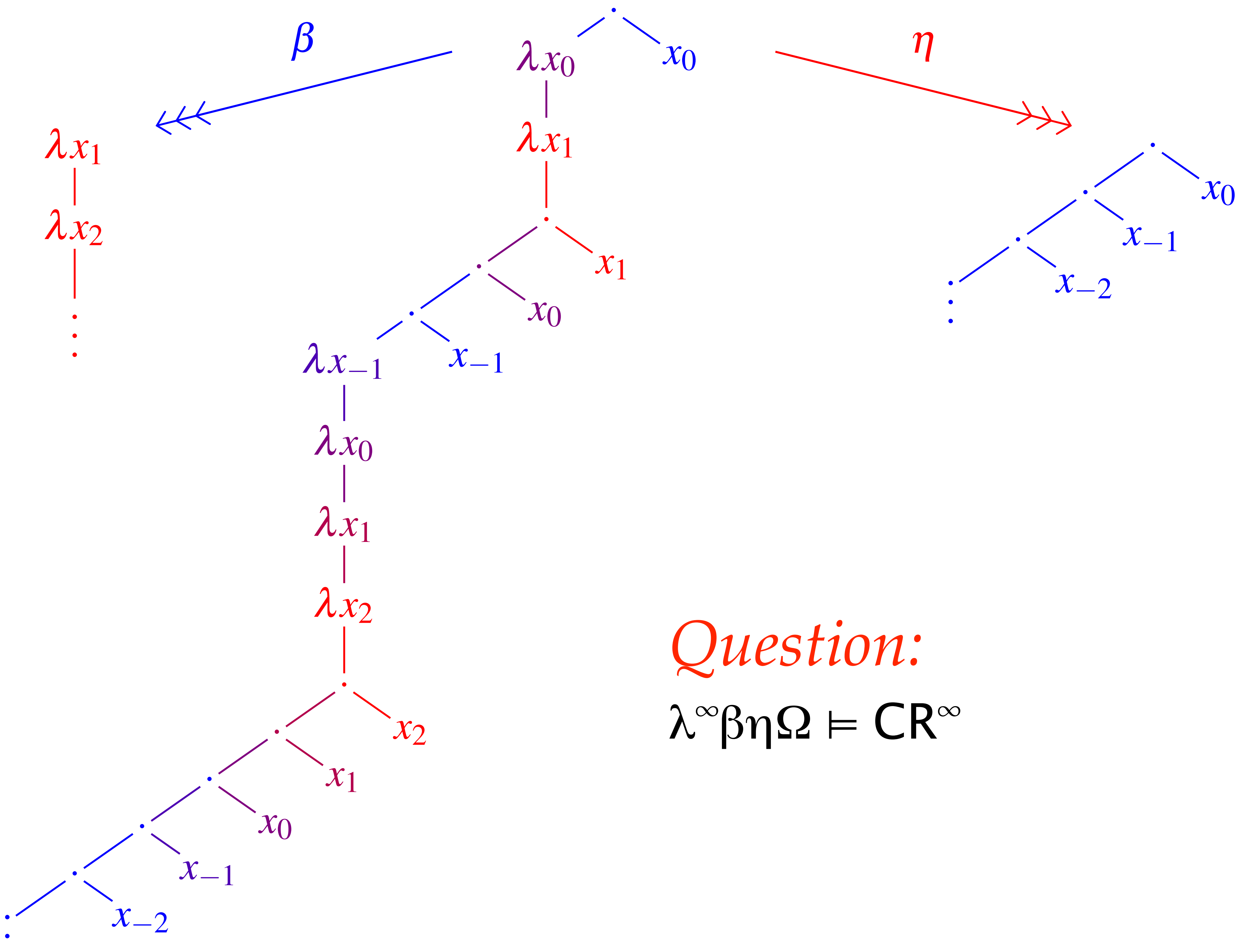
Now consider the term ψ defined as follows:

$$\psi = \mathsf{P}\;\mathsf{SS}\;\mathsf{PPP}\;\mathsf{SSSS}\;\mathsf{PPPPP}\;\mathsf{SSSSSS}\;\ldots$$

$$\mathsf{S}^\omega \twoheadleftarrow\!\!\!\twoheadleftarrow \psi \twoheadrightarrow\!\!\!\twoheadrightarrow \mathsf{P}^\omega$$



$\lambda^\infty\beta\eta \not\models \mathsf{UN}^\infty$



Question:

$\lambda^\infty\beta\eta\Omega \models \mathsf{CR}^\infty$

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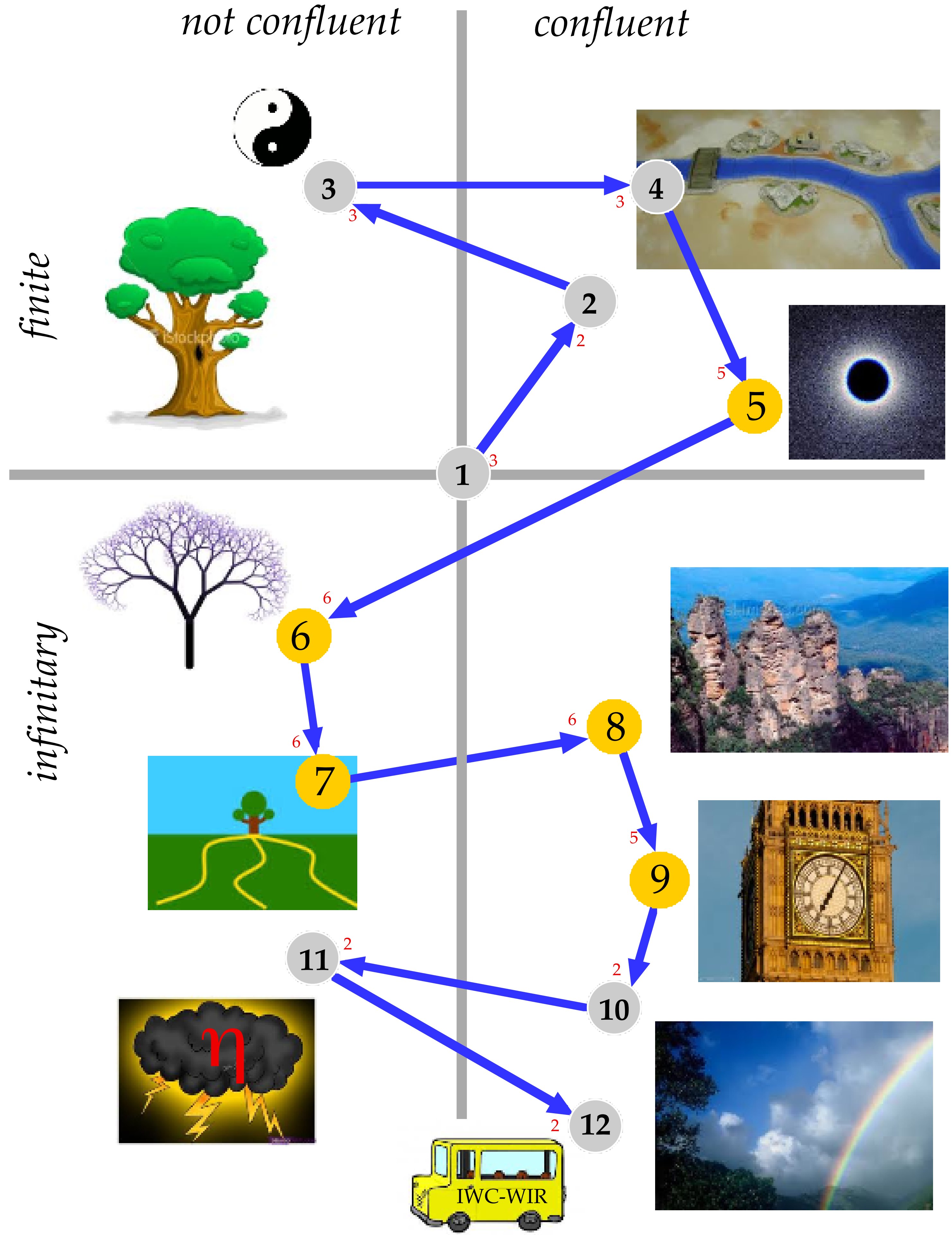
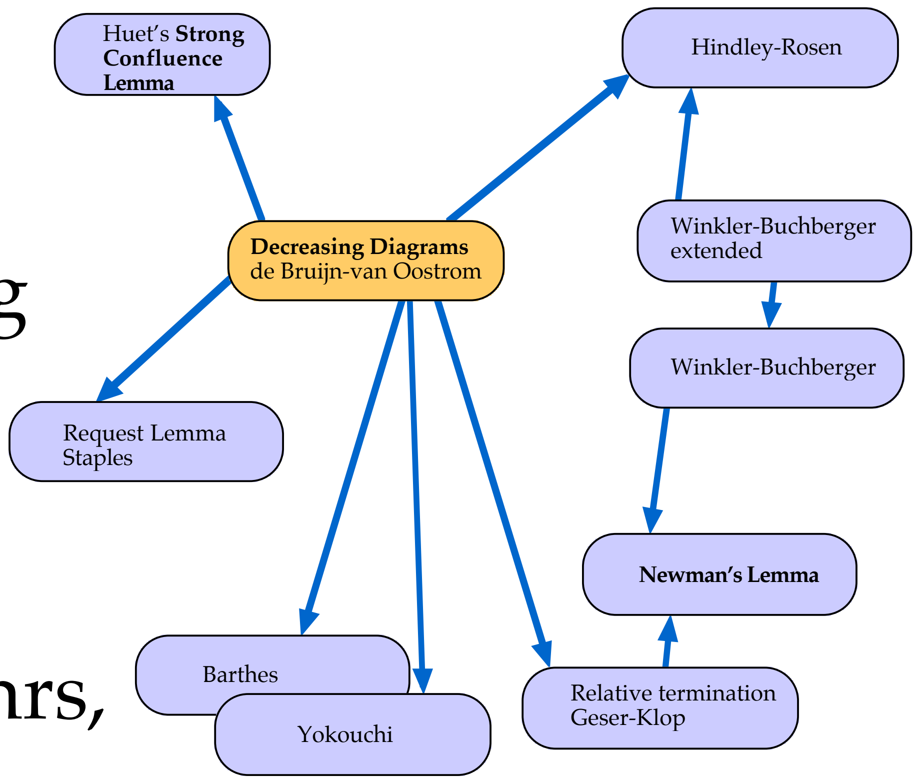


Other bus tours of the RTA-IWC-WIR company



Abstract rewriting and confluence, decreasing diagrams, De Bruijn - Van Oostrom

Alternative set-up of infinitary rewriting, Kahrs, ideal completion (Bahr), coinductive definition, Endrullis, Polonsky et al.

Infinitary Rewriting Coinductively

$$\twoheadrightarrow\!\!\!\!\rightarrow \;=\; \mu x.\, \nu y.\, (\rightarrow_\varepsilon \cup\, \overline{x})^* \circ \overline{y}$$

$$\overline{R} = \{\, \langle f(s_1,\ldots,s_n), f(t_1,\ldots,t_n)\rangle \mid s_1\, R\, t_1,\ldots,s_n\, R\, t_n \,\} \cup \mathsf{id}$$

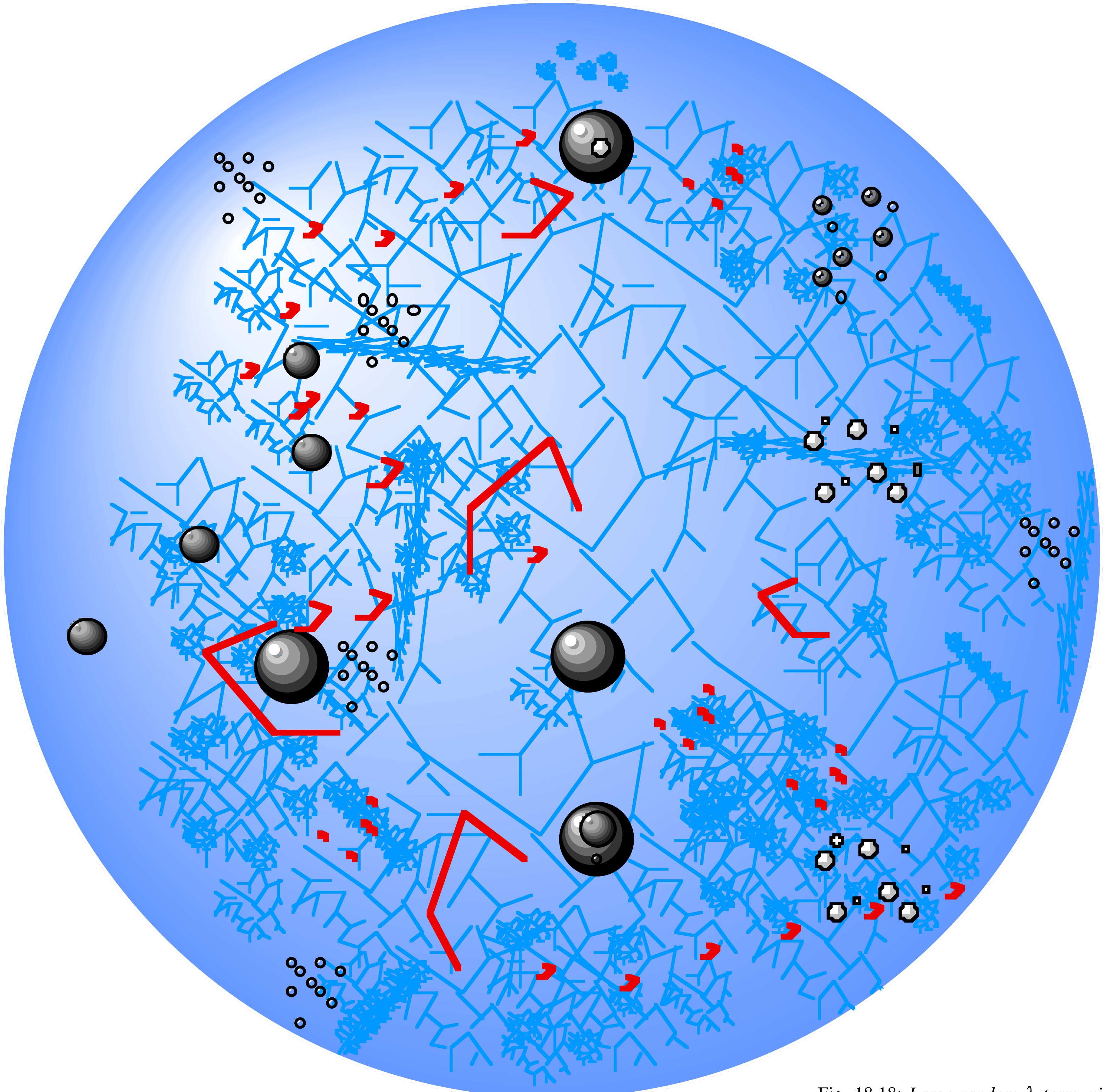


Fig. 18.18: *Large random λ-term, viewed as a mini-cosmos, evolving non-deterministically by local changes due to β-steps; their patterns are the red configurations. In the final result the place and nature of the normalized parts of the structure, as well as the singularities formed by the unsolvable terms, the black holes, is 'predestined', independent of the actual evolution path to the normal form, an infinite λβΩ-term.*

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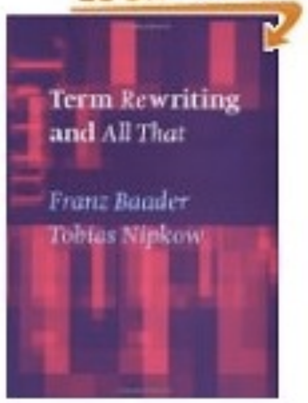

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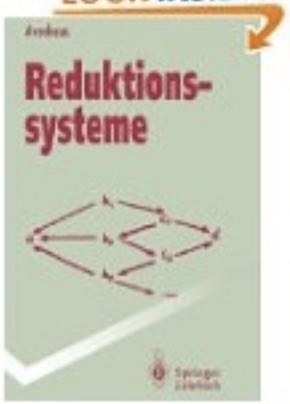

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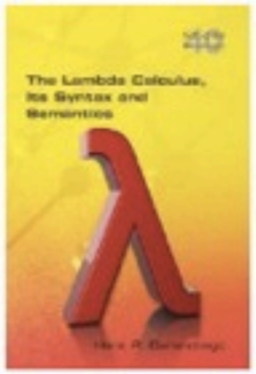

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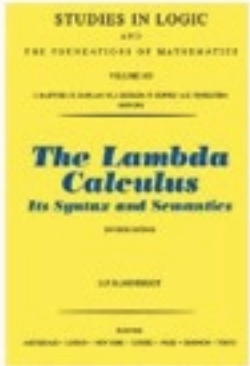

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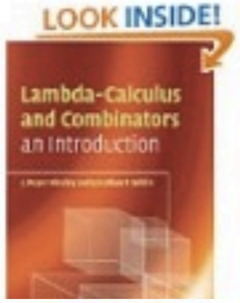

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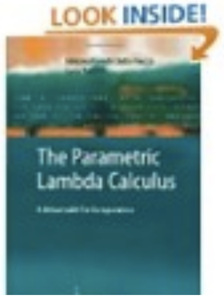

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S. Abramsky and C.-H. L. Ong.
Full Abstraction in the Lazy Lambda Calculus.
Information and Computation, 105(2):159–267, 1993.

Peter Aczel.
A general Church-Rosser theorem.
Technical report, University of Manchester, July 1978.

and and.
Highlights in Infinitary Rewriting and Lambda Calculus.
Theoretical Computer Science, 464:48–71, 2012.

Z. M. Ariola, R. Kennaway, J. W. Klop, M. R. Sleep, and F.-J. de Vries.
Syntactic Definitions of Undefined: On Defining the Undefined.
In *Proc. Conf. on Theoretical Aspects of Computer Software (TACS 1994)*, pages 543–554, 1994.

Jürgen Avenhaus.
Reduktionssysteme - Rechnen und Schliessen in gleichungsdefinierten Strukturen.
Springer-Lehrbuch. Springer, 1995.

F. Baader and T. Nipkow.

Term Rewriting and All That.
Cambridge University Press, 1998.

H. P. Barendregt.
The Type Free Lambda Calculus.
In Jon Barwise, editor, *Handbook of mathematical Logic*, pages 1091–1132. Nort-Holland Publishing Company, Amsterdam, 1977.

H. P. Barendregt.
The Lambda Calculus, its Syntax and Semantics, volume 103 of *Studies in Logic and The Foundations of Mathematics*.
North-Holland, revised edition, 1984.

H. P. Barendregt, J. Bergstra, J. W. Klop, and H. Volken.
Degrees, reductions and representability in the lambda calculus.
Technical report, Department of Mathematics, Utrecht University, 1976.

H. P. Barendregt, R. Kennaway, J. W. Klop, and M. R. Sleep.
Needed Reduction and Spine Strategies for the Lambda Calculus.
Information and Computation, 75(3):191–231, 1987.

H. P. Barendregt and J. W. Klop.
Applications of Infinitary Lambda Calculus.
Information of Computation, 207(5):559–582, 2009.

H.P. Barendregt, W. Dekkers, and R. Statman.
Lambda Calculus with Types.
Perspectives in Logic. Cambridge University Press, 2013.

A. Berarducci.
Infinite λ-Calculus and Non-Sensible Models.
In *Logic and Algebra (Pontignano, 1994)*, pages 339–377. Dekker, New York, 1996.

A. Berarducci and M. Dezani-Ciancaglini.
Infinite λ-calculus and types.
Theoretical Computer Science, 212(1-2):29–75, 1999.
Gentzen (Rome, 1996).

A. Berarducci and B. Intrigila.
Church–Rosser λ-theories, Infinite λ-calculus and Consistency Problems.
Logic: From Foundations to Applications, pages 33–58, 1996.

I. Bethke.
Lamda Calculus.
Chapter 10 in [97].

I. Bethke, J. W. Klop, and de Vrijer, R.C.

Descendants and Origins in Term Rewriting.
Information and Computation, 159(1–2):59–124, 2000.

M. Bezem, J. W. Klop, and V. van Oostrom.
Advanced ARS Theory.
In Terese, editor, *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*, pages 744–775. Cambridge University Press, 2003.

C. Böhm, editor.
Proc. Symp. on Lambda-Calculus and Computer Science Theory, volume 37 of *Lecture Notes in Computer Science*. Springer, 1975.

F. Cardone and J.R. Hindley.
History of lambda-calculus and combinatory logic.
In *Handbook of the History of Logic*. 2006.

A. Church and J.B. Rosser.
Some Properties of Conversion.
Transactions of the American Mathematical Society, 39:472–482, 1936.

Th. Coquand.
Infinite Objects in Type Theory.

In *Postproc. Conf. on Types for Proofs and Programs (TYPES 1993)*, volume 806 of *LNCS*, pages 62–78. Springer, 1993.

R. C. de Vrijer.
Surjective Pairing and Strong Normalization: Two Themes in Lambda Calculus.
PhD thesis, Universiteit van Amsterdam, 1987.

N. Dershowitz and J.-P. Jouannaud.
Rewrite systems.
In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, volume B: Formal Models and Semantics, chapter 15. North-Holland, Amsterdam, 1990.

N. Dershowitz, S. Kaplan, and D. A. Plaisted.
Rewrite, Rewrite, Rewrite, Rewrite, Rewrite, ...
Theoretical Computer Science, 83(1):71–96, 1991.

M. Dezani-Ciancaglini, P. Severi, and F.-J. de Vries.
Infinitary Lambda Calculus and Discrimination of Berarducci Trees.
Theoretical Computer Science, 2(298):275–302, 2003.

R. Di Cosmo and D. Kesner.
Simulating expansions without expansions.
Mathematical Structures in Computer Science, 4:1–48, 1994.

J. Endrullis and R. de Vrijer.
Reduction Under Substitution.
In *Proc. 19th Int. Conf. on Rewriting Techniques and Applications (RTA 2008)*, volume 5117 of *Lecture Notes in Computer Science*, pages 425–440. Springer, 2008.

J. Endrullis, H. Geuvers, J. G. Simonsen, and H. Zantema.
Levels of Undecidability in Rewriting.
Information and Computation, 209(2):227–245, 2011.

J. Endrullis, C. Grabmayer, D. Hendriks, A. Isihara, and J. W. Klop.
Productivity of Stream Definitions.
Theoretical Computer Science, 411:765–782, 2010.

J. Endrullis, C. Grabmayer, D. Hendriks, J. W. Klop, and R. C. de Vrijer.
Proving Infinitary Normalization.
In *Postproc. Int. Workshop on Types for Proofs and Programs (TYPES 2008)*, volume 5497 of *Lecture Notes in Computer Science*, pages 64–82. Springer, 2009.

J. Endrullis, C. Grabmayer, D. Hendriks, J. W. Klop, and V. van Oostrom.
Unique Normal Forms in Infinitary Weakly Orthogonal Rewriting.

In *Proc. 21st Int. Conf. on Rewriting Techniques and Applications (RTA 2010)*, volume 6 of *Leibniz International Proceedings in Informatics*, pages 85–102. Schloss Dagstuhl, 2010.

J. Endrullis and D. Hendriks.
Lazy Productivity via Termination.
Theoretical Computer Science, 412(28):3203–3225, 2011.

J. Endrullis, D. Hendriks, and J. W. Klop.
Modular Construction of Fixed Point Combinators and Clocked Böhm Trees.
In *Proc. Symp. on Logic in Computer Science (LICS 2010)*, pages 111–119. IEEE Computer Society, 2010.

J. Endrullis, D. Hendriks, and J. W. Klop.
Clocked lambda calculus.
2013.
To appear.

J. Endrullis and A. Polonsky.
A Characterization of Reduction 1-Cycles in Infinitary Lambda Calculus.
In *TYPES*, 2012.

Jörg Endrullis, Clemens Grabmayer, Jan Willem Klop, and Vincent van Oostrom.
On equal -terms.
Theor. Comput. Sci., 412(28):3175–3202, 2011.

Jörg Endrullis, Helle Hvid Hansen, Dimitri Hendriks, Andrew Polonsky, and Alexandra Silva.
A coinductive treatment of infinitary rewriting.
CoRR, abs/1306.6224, 2013.

Enno Folkerts.
Invertibility in lambda-eta.
In *LICS*, pages 418–429. IEEE, 1998.

C. Hankin.
Lambda Calculi: A Guide for Computer Scientists.
Oxford University Press, 1994.

J. R. Hindley and P. Seldin.
Lambda-Calculus and Combinators.
Cambridge University Press, 2008.

S. Kahrs.
Infinitary Rewriting: Foundations Revisited.

In *Proc. 21st Int. Conf. on Rewriting Techniques and Applications (RTA 2010)*, volume 6 of *Leibniz International Proceedings in Informatics*, pages 161–176. Schloss Dagstuhl, 2010.

J. R. Kennaway.
On transfinite abstract reduction systems.
Technical Report CS-R9205, CWI, January 1992.

J. R. Kennaway and F.-J. de Vries.
Infinitary Rewriting, chapter 12.
Cambridge University Press, 2003.
in [97].

J. R. Kennaway, J. W. Klop, M. R. Sleep, and F.-J. de Vries.
An Infinitary Church–Rosser Property for Non-Collapsing Orthogonal Term Rewriting Systems.
Technical Report CS-R9043, CWI, 1990.

J. R. Kennaway, J. W. Klop, M. R. Sleep, and F.-J. de Vries.
From Finite to Infinite Lambda Calculi.
In Hiroakira Ono, editor, *Special issue dedicated to the Workshop on Non-standard Logics and Logical Aspects of Computer Science, 1994*, volume 24 of *Bulletin of the section of Logic*, pages 13–20. University of Lodz, Department of Logic, 1995.

J. R. Kennaway, J. W. Klop, M. R. Sleep, and F.-J. de Vries.
Infinitary Lambda Calculi and Böhm Models.
In *Proc. Conf. on Rewriting Techniques and Applications (RTA 1995)*, pages 257–270, 1995.

J. R. Kennaway, J. W. Klop, M. R. Sleep, and F.-J. de Vries.
Transfinite Reductions in Orthogonal Term Rewriting Systems.
Information and Computation, 119(1):18–38, 1995.

J. R. Kennaway, J. W. Klop, M. R. Sleep, and F.-J. de Vries.
Infinitary Lambda Calculus.
Theoretical Computer Science, 175(1):93–125, 1997.

J. R. Kennaway, V. van Oostrom, and F.-J. de Vries.
Meaningless Terms in Rewriting.
The Journal of Functional and Logic Programming, 1, 1999.

J. Ketema.
Böhm-Like Trees for Rewriting.
PhD thesis, Vrije Universiteit Amsterdam, 2006.

J. Ketema.
On Normalisation of Infinitary Combinatory Reduction Systems.

In *Proc. 19th Int. Conf. on Rewriting Techniques and Applications (RTA 2008)*, volume 5117 of *Lecture Notes in Computer Science*, pages 172–186. Springer, 2008.

J. Ketema.
Comparing Böhm-Like Trees.
In *Proc. 20th Int. Conf. on Rewriting Techniques and Applications (RTA 2009)*, volume 5595 of *Lecture Notes in Computer Science*, pages 239–254. Springer, 2009.

J. Ketema.
Counterexamples in Infinitary Rewriting with Non-fully-extended Rules.
Information Processing Letters, 111(13):642–646, 2011.

J. Ketema.
Reinterpreting Compression in Infinitary Rewriting.
In *Proc. 23rd Int. Conf. on Rewriting Techniques and Applications (RTA 2012)*, volume 15 of *Leibniz International Proceedings in Informatics*, pages 209–224. Schloss Dagstuhl, 2012.

J. Ketema and J. G. Simonsen.
Infinitary Combinatory Reduction Systems.

In *Proc. 16th Int. Conf. on Rewriting Techniques and Applications (RTA 2005)*, volume 3467 of *Lecture Notes in Computer Science*, pages 438–452. Springer, 2005.

J. Ketema and J. G. Simonsen.
Infinitary Combinatory Reduction Systems: Confluence.
Logical Methods in Computer Science, 5(4):1–29, 2009.

J. Ketema and J. G. Simonsen.
Infinitary Combinatory Reduction Systems: Normalising Reduction Strategies.
Logical Methods in Computer Science, 6(1:7):1–35, 2010.

J. Ketema and J. G. Simonsen.
Infinitary Combinatory Reduction Systems.
Information and Computation, 209(6):893–926, 2011.

J. W. Klop.
Combinatory Reduction Systems, volume 127 of *Mathematical centre tracts*.
Mathematisch Centrum, 1980.

J. W. Klop.
Term Rewriting Systems.

In *Handbook of Logic in Computer Science*, volume II, pages 1–116. Oxford University Press, 1992.

J. W. Klop.
New Fixed Point Combinators from Old.
In *Reflections on Type Theory, λ-Calculus, and the Mind. Essays Dedicated to Henk Barendregt on the Occasion of his 60th Birthday*, pages 197–210. 2007.

J. W. Klop and R. C. de Vrijer.
Extended term rewriting systems.
In S. Kaplan and M. Okada, editors, *Conditional and typed rewriting systems*, volume 516 of *Lecture Notes in Computer Science*, pages 26–50. Springer, Berlin, 1991.

J. W. Klop and R. C. de Vrijer.
Infinitary Normalization.
In *We Will Show Them: Essays in Honour of Dov Gabbay (2)*, pages 169–192. College Publications, 2005.

J. W. Klop, V. van Oostrom, and F. van Raamsdonk.
Combinatory reduction systems: introduction and survey.
Theoretical Computer Science, 121(1 & 2):279–308, 1993.

J. W. Klop, V. van Oostrom, and F. van Raamsdonk.

Liber Amicorum for Roel de Vrijer, Letters and essays on the occasion of his 60th birthday, 2009.

Jan Willem Klop and Roel C. de Vrijer.
Unique normal forms for lambda calculus with surjective pairing.
Inf. Comput., 80(2):97–113, 1989.

A. Kurz, D. Petrisan, P. Severi, and F.-J. de Vries.
An Alpha-Corecursion Principle for the Infinitary Lambda Calculus.
In *Proc. 11th Int. Workshop on Coalgebraic Methods in Computer Science*. Springer, 2012.

B. Lercher.
Lambda-Calculus Terms That Reduce To Themselves.
Notre Dame Journal of Formal Logic, 17(2):291–292, 1976.

J.-J. Lévy.
An Algebraic Interpretation of the $\lambda\beta K$-Calculus, and an Application of a Labelled λ-Calculus.
Theoretical Computer Science, 2(1):97–114, 1976.

S. Lucas.
Transfinite Rewriting Semantics for Term Rewriting Systems.

In *Proc. 12th Conf. on Rewriting Techniques and Applications (RTA 2001)*, volume 2051 of *LNCS*, pages 216–230. Springer, 2001.

R. Mayr and T. Nipkow.
Higher-order rewrite systems and their confluence.
Theoretical Computer Science, 192:3–29, 1998.

A. Middeldorp.
Call by Need Computations to Root-Stable Form.
In *Proc. Symp. on Principles of Programming Languages (POPL 1997)*, pages 94–105. ACM, 1997.

A. Middeldorp, V. van Oostrom, F. van Raamsdonk, and R. C. de Vrijer, editors.
Processes, Terms and Cycles: Steps on the Road to Infinity, Essays Dedicated to Klop, J. W., on the Occasion of his 60th Birthday, volume 3838 of *LNCS*. Springer, 2005.

R. Nakajima.
Infinite normal forms for the λ-calculus.
In C. Böhm, editor, *Lambda calculus and Computer Science Theory*, volume 37 of *LNCS*, pages 62–82. Springer-Verlag, 1975.

E. Ohlebusch.
Advanced Topics in Term Rewriting.

Springer, 2002.

C.-H. L. Ong.
The Lazy Lambda Calculus: an Investigation into the Foundations of Functional Programming.
PhD thesis, University of Cambridge, 1992.

V. van Oostrom.
Confluence for Abstract and Higher-Order Rewriting.
PhD thesis, Vrije Universiteit Amsterdam, 1994.

V. van Oostrom.
Higher-order families.
In *Proceedings of the 7th International Conference on Rewriting Techniques and Applications (RTA '96)*, volume 1103 of *Lecture Notes in Computer Science*, pages 392–407. Springer-Verlag, 1996.

V. van Oostrom.
Normalisation in Weakly Orthogonal Rewriting.
In *Proc. 10th Intl. Conf. on Rewriting Techniques and Applications (RTA 1999)*, volume 1631 of *LNCS*, pages 60–74. Springer, 1999.

S. Ronchi Della Rocca.
The Parametric Lambda Calculus: A Metamodel for Computation.
Springer, 2004.

A. Salibra.
Topological incompleteness and order incompleteness of the lambda calculus.
ACM Transactions on Computational Logic, 4(3):379–401, 2003. (Special Issue LICS 2001).

D. Sangiorgi.
The lazy lambda calculus in a concurrency scenario.
Information and Computation, 111(1):120–153, 1994.

D. S. Scott.
Some Philosophical Issues Concerning Theories of Combinators.
In *Lambda Calculus and Computer Science Theory*, volume 37 of *Lecture Notes in Computer Science*, pages 346–366, 1975.

P. Severi and F.-J. de Vries.
A Lambda Calculus for D_∞.
Technical report, University of Leicester, 2002.

P. Severi and F.-J. de Vries.
An Extensional Böhm Model.
In *Rewriting Techniques and Applications*, volume 2378 of *LNCS*, pages 159–173. Springer, 2002.

P. Severi and F.-J. de Vries.
Continuity and Discontinuity in Lambda Calculus.
In *Typed Lambda Calculus and Applications*, volume 3461 of *LNCS*, pages 369–385. Springer, 2005.

P. Severi and F.-J. de Vries.
Order Structures for Böhm-like Models.
In *Computer Science Logic*, volume 3634 of *LNCS*. Springer, 2005.

P. Severi and F.-J. de Vries.
Decomposing the Lattice of Meaningless Sets in the Infinitary Lambda Calculus.
In *Proc. of 18th Int. Workshop on Logic, Language, Information and Computation (WoLLIC 2011)*, volume 6642 of *Lecture Notes in Computer Science*, pages 210–227. Springer, 2011.

P. Severi and F.-J. de Vries.
Weakening the Axiom of Overlap in Infinitary Lambda Calculus.
In *Proc. 22nd Int. Conf. on Rewriting Techniques and Applications (RTA 2011)*, volume 10 of *Leibniz International Proceedings in Informatics*, pages 313–328. Schloss Dagstuhl, 2011.

P. Severi and F.-J. de Vries.
Meaningless Sets in Infinitary Combinatory Logic.

In *Proc. 23rd Int. Conf. on Rewriting Techniques and Applications (RTA 2012)*, volume 15 of *Leibniz International Proceedings in Informatics*, pages 288–304. Schloss Dagstuhl, 2012.

J. G. Simonsen.
On Confluence and Residuals in Cauchy Convergent Transfinite Rewriting.
Information Processing Letters, 91(3):141–146, 2004.

J. G. Simonsen.
Weak Convergence and Uniform Normalization in Infinitary Rewriting.
In *Proc. 20th Int. Conf. on Rewriting Techniques and Applications (RTA 2009)*, volume 6 of *Leibniz International Proceedings in Informatics*, pages 311–324. Schloss Dagstuhl, 2010.

R. Smullyan.
To Mock a Mockingbird, and Other Logic Puzzles.
Alfred A. Knopf, New York, 1985.

R. Statman.
The Word problem for Smullyan's Lark Combinator is Decidable.
Journal of Symbolic Computation, 7:103–112, 1989.

Richard Statman.

Every countable poset is embeddable in the poset of unsolvable terms.
Theor. Comput. Sci., 48(3):95–100, 1986.

Terese.
Term Rewriting Systems, volume 55 of *Cambridge Tracts in Theoretical Computer Science*.
Cambridge University Press, 2003.

V. van Oostrom and F. van Raamsdonk.
Weak orthogonality implies confluence: the higher-order case.
In *Proceedings of the Third International Symposium on Logical Foundations of Computer Science (LFCS '94)*, volume 813 of *Lecture Notes in Computer Science*, pages 379–392. Springer, 1994.

F. van Raamsdonk.
Higher-order rewriting.
In *Term Rewriting Systems*, Chapter 11, pages 588–667.

F.-J. de Vries.
Böhm Trees, Bisimulations and Observations in Lambda Calculus.
In *Proc. 2nd Fuji Int. Workshop on Functional and Logic Programming*, pages 230–245. World Scientific, 1997.

C. P. Wadsworth.

The relation between computational and denotational properties for Scott's D_∞-models of the lambda-calculus.
SIAM Journal on Computing, 5(3):488–521, 1976.

C. P Wadsworth.
Approximate reduction and lambda calculus models.
SIAM Journal on Computing, 7(3):337–356, 1978.

J. Waldmann.
The Combinator S.
Information and Computation, 159(1–2):2–21, 2000.

H. Zantema.
Normalization of Infinite Terms.
In *Proc. 19th Int. Conf. on Rewriting Techniques and Applications (RTA 2008)*, number 5117 in Lecture Notes in Computer Science, pages 441–455, 2008.

H. Zantema.
Well-Definedness of Streams by Transformation and Termination.
Logical Methods in Computer Science, 6(3), 2010.

H. Zantema and M. Raffelsieper.
Proving Productivity in Infinite Data Structures.

In *Proc. 21st Int. Conf. on Rewriting Techniques and Applications (RTA 2010)*, volume 6 of *Leibniz International Proceedings in Informatics*, pages 401–416. Schloss Dagstuhl, 2010.