Proving Confluence of Conditional Term Rewriting Systems via Unravelings

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Example: Conditional TRS even/odd

Example (CTRS representing Even/Odd)

$$\mathcal{R}_{\mathsf{even}} = \left\{ \begin{array}{r} \mathsf{even}(0) \to \mathsf{true} \\ \mathsf{odd}(0) \to \mathsf{false} \\[1ex] \mathsf{even}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{odd}(x) \to^{*} \mathsf{true} \\ \mathsf{even}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{odd}(x) \to^{*} \mathsf{false} \\[1ex] \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{even}(x) \to^{*} \mathsf{true} \\ \mathsf{odd}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{even}(x) \to^{*} \mathsf{false} \end{array} \right\}$$

Outline

Conditional term rewriting

CTRSs

- Conditional rule: $l \to r \Leftarrow s_1 = t_1, \ldots, s_k = t_k$
- Oriented CTRS: "$=$" is interpreted as "$\to^*$"
- Conditions cause problems

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- Non-overlapping
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$$f(a) \nearrow c \qquad f(a) \searrow f(b)$$

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Transformation of CTRSs into unconditional TRSs

Sequential unraveling of [Ohlebusch 2002]

$$l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_k \to^* t_k \implies \left\{ \begin{array}{rcl} l & \to & U_1(s_1, \vec{X}_1) \\ U_1(t_1, \vec{X}_1) & \to & U_2(s_2, \vec{X}_2) \\ \vdots & & \vdots \\ U_k(t_k, \vec{X}_k) & \to & r \end{array} \right.$$

- $X_i = \mathcal{V}ar(l, s_1, t_1, \ldots, s_{i-1}, t_{i-1})$
- Used to prove properties of CTRSs like (operational) termination, so why not confluence?

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How to prove confluence of CTRS $\mathcal{R}$ via $\mathbb{U}(\mathcal{R})$?

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- $s \to_{\mathbb{U}(\mathcal{R})} t \;\Rightarrow\; s \to_{\mathcal{R}} \mathrm{tb}(t)$ for weakly left-linear CTRSs.
- Implies soundness for joinability $s \downarrow_{\mathbb{U}(\mathcal{R})} t \;\Rightarrow\; s \downarrow_{\mathcal{R}} t$
- CR of $\mathbb{U}(\mathcal{R})$ + Soundness for joinability $\Rightarrow$ CR of $\mathcal{R}$.

New proof for result of [Suzuki, Middeldorp, Ida, RTA 1995]

- Orthogonal properly oriented right-stable 3-CTRSs are confluent.

The unraveling $\mathbb{U}_{conf}$

Example (even/odd-CTRS)

$$\mathsf{even}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{odd}(x) \to^* \mathsf{true} \qquad \begin{cases} \mathsf{even}(\mathsf{s}(x)) \to U_1(\mathsf{odd}(x), x) \\ U_1(\mathsf{true}, x) \to \mathsf{true} \end{cases}$$

$$\mathsf{even}(\mathsf{s}(x)) \to \mathsf{false} \Leftarrow \mathsf{odd}(x) \to^* \mathsf{false} \qquad \begin{cases} \mathsf{even}(\mathsf{s}(x)) \to U_2(\mathsf{odd}(x), x) \\ U_2(\mathsf{false}, x) \to \mathsf{false} \end{cases}$$

$$\mathit{even}(\mathsf{s}(0)) \begin{array}{l} \nearrow U_1(\mathsf{odd}(0), 0) \to U_1(\mathsf{false}, 0) \\ \searrow U_2(\mathsf{odd}(0), 0) \to U_2(\mathsf{false}, 0) \to \mathsf{false} \end{array}$$

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- Idea: Pick labels for U-symbols based on terms in rule

$$l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_k \to^* t_k \qquad \begin{cases} l \to U_{l,s_1}(s_1, \vec{X}_1) \\ U_{l,s_1}(t_1, \vec{X}_1) \to U_{l,s_1,t_1,t_2}(s_2, \vec{X}_2) \\ \quad \vdots \qquad \vdots \\ U_{l,s_1,t_1,\ldots,s_k}(t_k, \vec{X}_k) \to r \end{cases}$$

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Example (even/odd-CTRS using $\mathbb{U}_{conf}$)

$$\mathsf{even}(\mathsf{s}(x)) \to \mathsf{true} \Leftarrow \mathsf{odd}(x) \to^* \mathsf{true} \left\{ \begin{array}{l} \mathsf{even}(\mathsf{s}(x)) \to U_{even(s(x)),odd(x)}(\mathsf{odd}(x), x) \\ U_{even(s(x)),odd(x)}(\mathsf{true}, x) \to \mathsf{true} \end{array} \right.$$

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even/odd-CTRS

Example

$$\mathbb{U}_{conf}(\mathcal{R}) = \left\{ \begin{array}{rcl} \mathsf{even}(0) & \rightarrow & \mathsf{true} \\ \mathsf{even}(\mathsf{s}(x)) & \rightarrow & U_{even(s(x)),odd(x)}(\mathsf{odd}(x), x) \\ U_{even(s(x)),odd(x)}(\mathsf{true}, x) & \rightarrow & \mathsf{true} \\ U_{even(s(x)),odd(x)}(\mathsf{false}, x) & \rightarrow & \mathsf{false} \\ \mathsf{odd}(0) & \rightarrow & \mathsf{false} \\ \mathsf{odd}(\mathsf{s}(x)) & \rightarrow & U_{odd(s(x)),even(x)}(\mathsf{even}(x), x) \\ U_{odd(s(x)),even(x)}(\mathsf{true}, x) & \rightarrow & \mathsf{true} \\ U_{odd(s(x)),even(x)}(\mathsf{false}, x) & \rightarrow & \mathsf{false} \end{array} \right\}$$

- $\mathbb{U}_{conf}(\mathcal{R})$ is left-linear $\Longrightarrow$ $\mathbb{U}_{conf}$ is sound for joinability
- $\mathbb{U}_{conf}(\mathcal{R})$ is SN and non-overlapping $\Longrightarrow$ $\mathbb{U}_{conf}(\mathcal{R})$ is confluent
- $\Longrightarrow$ $\mathcal{R}$ is confluent.

Conclusion and Perspectives

What we have shown

- Unravelings can be used to prove confluence of CTRSs
- The unraveling $\mathbb{U}_{conf}$ has good properties for overlay CTRSs.

What we might show in the future

- Reachability analysis using tree automata
- Different transformations
- Tools proving confluence of CTRSs