Disproving Confluence of Term Rewriting Systems

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IWC 2013

Outline

1. Backgrounds
2. Proving Non-Joinability by Interpretation
3. Proving Non-Joinability by Ordering
4. Implementation and Experiments

Disproving Confluence of TRSs

Find terms t_1, t_2 such that

(1) $s \xrightarrow{*} t_1$ and $s \xrightarrow{*} t_2$ for some s, and
(finding 'candidates' for non-confluence witness)

(2) $t_1 \xrightarrow{*} u$ and $t_2 \xrightarrow{*} u$ for no u,
i.e. $\{u \mid t_1 \xrightarrow{*} u\} \cap \{v \mid t_2 \xrightarrow{*} v\} = \emptyset$.
(proving non-joinability of 'candidates')

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We let the problem (1) untouched, and consider the problem (2).

We abbreviate $\{u \mid t_1 \xrightarrow{*} u\} \cap \{v \mid t_2 \xrightarrow{*} v\} = \emptyset$ as $\mathrm{NJ}(t_1, t_2)$.

Proving Non-Joinability by Tree Automata

Only(?) serious approach for proving non-joinability is using tree automata approximation (Durand-Middeldorp, CADE 1997; Genet, RTA 1998).

(1) Construct tree automata $\mathcal{A}_1, \mathcal{A}_2$ such that $\{u \mid t_i \xrightarrow{*} u\} \subseteq \mathcal{L}(\mathcal{A}_i)$ $(i = 1, 2)$ by tree automata approximation.

(2) Check $\mathcal{L}(\mathcal{A}_1) \cap \mathcal{L}(\mathcal{A}_2) = \emptyset$.

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Sometimes it is difficult to construct a well-approximated tree automaton.

This work: another approach for proving non-joinability.

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Interpretation

We first recall some standard definitions.

An $\mathcal{F}$-algebra $\mathcal{A} = \langle A, \langle f^{\mathcal{A}} \rangle_{f \in \mathcal{F}} \rangle$ is a set A equipped with functions $f^{\mathcal{A}} : A^n \to A$ for each n-ary function symbol $f \in \mathcal{F}$.

A valuation σ on a $\mathcal{F}$-algebra $\mathcal{A}$ is a mapping $\sigma : \mathcal{V} \to A$.

The interpretation $[\![t]\!]_\sigma \in A$ of a term $t \in \mathrm{T}(\mathcal{F}, \mathcal{V})$ is given by

$$[\![x]\!]_\sigma = \sigma(x)$$
$$[\![f(t_1, \ldots, t_n)]\!]_\sigma = f^{\mathcal{A}}([\![t_1]\!]_\sigma, \ldots, [\![t_n]\!]_\sigma)$$

Idea of Using Interpretation

If there exist an $\mathcal{F}$-algebra and a valuation σ such that
(i) $u \to_{\mathcal{R}} v$ implies $[\![u]\!]_\sigma = [\![v]\!]_\sigma$ and (ii) $[\![s]\!]_\sigma \neq [\![t]\!]_\sigma$,
then $\mathrm{NJ}(s, t)$.

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Idea: replace (i) by the following (i′)

(i′) $u \to_{\{l \to r\}} v$ implies $[\![u]\!]_\sigma = [\![v]\!]_\sigma$ for any usable rule $l \to r \in \mathcal{R}$.

Here, *usable* means it can happen $s \xrightarrow{*}_{\mathcal{R}} \circ \to_{\{l \to r\}} u$ or $t \xrightarrow{*}_{\mathcal{R}} \circ \to_{\{l \to r\}} u$ for some u (given in the next slide).

Usable Rules for Non-Joinability

Definition. The set of **usable rules** $\mathcal{U}(s) \subseteq \mathcal{R}$ is the smallest set satisfying:
(i) for any non-variable subterm $f(u_1, \ldots, u_n)$ of s and $l \to r \in \mathcal{R}$, if $f(\mathrm{TCAP}(u_1), \ldots, \mathrm{TCAP}(u_n))$ and l are unifiable then $l \to r \in \mathcal{U}(s)$; and
(ii) if $l' \to r' \in \mathcal{U}(s)$ and $l \to r \in \mathcal{U}(r')$, then $l \to r \in \mathcal{U}(s)$.

Lemma. If $s \xrightarrow{*}_{\mathcal{R}} \circ \to_{\{l \to r\}} t$ then $l \to r \in \mathcal{U}(s)$.

Here, we assume variable conditions of rewrite rules. It is straightforward to generalize usable rules to the case variable conditions do not hold.

Non-Joinability by Interpretation

Theorem 1. Let $\mathcal{A} = \langle A, \langle f^{\mathcal{A}} \rangle_{f \in \mathcal{F}} \rangle$ be an $\mathcal{F}$-algebra with $A = \biguplus_{i \in I} A_i$, and s, t terms. Suppose
(i) $[\![l]\!]_\sigma \in A_i$ implies $[\![r]\!]_\sigma \in A_i$ for any $l \to r \in \mathcal{U}(s) \cup \mathcal{U}(t)$,
(ii) if $a \in A_i$ implies $f^{\mathcal{A}}(\ldots, a, \ldots) \in A_j$, then for any $b \in A_i$, $f^{\mathcal{A}}(\ldots, b, \ldots) \in A_j$, and
(iii) $[\![s]\!]_\rho \in A_i$ and $[\![t]\!]_\rho \in A_j$ with $i \neq j$ for some ρ.
Then $\mathrm{NJ}(s, t)$.

(Proof Sketch) (i),(ii) imply that for any $s \xrightarrow{*}_{\mathcal{R}} u \to_{\mathcal{R}} v$, $[\![u]\!]_\rho \in A_i$ implies $[\![v]\!]_\rho \in A_i$. □

Example 1.

$$\mathcal{R} = \left\{ \begin{array}{llll} (1) & \mathsf{a} \to \mathsf{h}(\mathsf{c}) & (3) & \mathsf{h}(x) \to \mathsf{h}(\mathsf{h}(x)) \\ (2) & \mathsf{a} \to \mathsf{h}(\mathsf{f}(\mathsf{c})) & (4) & \mathsf{f}(x) \to \mathsf{f}(\mathsf{g}(x)) \end{array} \right\}.$$

Take candidates $\mathsf{h}(\mathsf{c})$, $\mathsf{h}(\mathsf{f}(\mathsf{c}))$. **Usable rules are** $\{(3), (4)\}$.

Take an $\mathcal{F}$**-algebra** $\mathcal{A} = \langle \{0, 1\}, \langle f^{\mathcal{A}} \rangle_{f \in \mathcal{F}} \rangle$ **as**

$\mathsf{a}^{\mathcal{A}} = \mathsf{c}^{\mathcal{A}} = 0$,
$\mathsf{f}^{\mathcal{A}}(n) = 1 - n$,
$\mathsf{h}^{\mathcal{A}}(n) = \mathsf{g}^{\mathcal{A}}(n) = n$.

Then $[\![\mathsf{h}(x)]\!]_\sigma = [\![\mathsf{h}(\mathsf{h}(x))]\!]_\sigma$, $[\![\mathsf{f}(x)]\!]_\sigma = [\![\mathsf{f}(\mathsf{g}(x))]\!]_\sigma$ **and** $[\![\mathsf{h}(\mathsf{c})]\!] \neq [\![\mathsf{h}(\mathsf{f}(\mathsf{c}))]\!]$. **Hence,** $\mathrm{NJ}(\mathsf{h}(\mathsf{c}), \mathsf{h}(\mathsf{f}(\mathsf{c})))$.

Example 2.

$$\mathcal{R} = \left\{ \begin{array}{llll} (1) & \mathsf{a} \to \mathsf{f}(\mathsf{c}) & (3) & \mathsf{f}(x) \to \mathsf{h}(\mathsf{g}(x)) \\ (2) & \mathsf{a} \to \mathsf{h}(\mathsf{c}) & (4) & \mathsf{h}(x) \to \mathsf{f}(\mathsf{g}(x)) \end{array} \right\}.$$

Take candidates $\mathsf{f}(\mathsf{c})$ **and** $\mathsf{h}(\mathsf{c})$**. Usable rules are** $\{(3), (4)\}$**.**

Take an $\mathcal{F}$**-algebra** $\mathcal{A} = \langle \mathbb{N}, \langle f^{\mathcal{A}} \rangle_{f \in \mathcal{F}} \rangle$ **as**

$\mathsf{a}^{\mathcal{A}} = \mathsf{c}^{\mathcal{A}} = 0$
$\mathsf{g}^{\mathcal{A}}(n) = n + 1$
$\mathsf{f}^{\mathcal{A}}(n) = n$
$\mathsf{h}^{\mathcal{A}}(n) = n + 1$

Then $[\![\mathsf{f}(x)]\!]_\sigma \equiv [\![\mathsf{h}(\mathsf{g}(x))]\!]_\sigma \pmod 2$, $[\![\mathsf{h}(x)]\!]_\sigma \equiv [\![\mathsf{f}(\mathsf{g}(x))]\!]_\sigma \pmod 2$ **and** $[\![\mathsf{f}(\mathsf{c})]\!] \not\equiv [\![\mathsf{h}(\mathsf{c})]\!] \pmod 2$**. Hence** $\mathrm{NJ}(\mathsf{f}(\mathsf{c}), \mathsf{h}(\mathsf{c}))$**.**

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Non-Joinability by Ordered $\mathcal{F}$-algebras

For a set of integers, an obvious choice of partition is $A = \{n \in A \mid n < k\} \uplus \{n \in A \mid k \leq n\}$ for some fixed k. **More generally, one can use** ordered $\mathcal{F}$-algebras $\mathcal{A} = \langle A, \leq, \langle f^{\mathcal{A}} \rangle_{f \in \mathcal{F}} \rangle$, **where $\leq$ is a partial order on A.**

Theorem 2. Let $\mathcal{A}$ be a weakly monotone ordered $\mathcal{F}$-algebra and s, t be terms. Suppose
(i) $[\![l]\!]_\sigma \leq [\![r]\!]_\sigma$ for any valuation σ and any $l \to r \in \mathcal{U}(s)$,
(ii) $[\![l]\!]_\sigma \geq [\![r]\!]_\sigma$ for any valuation σ and any $l \to r \in \mathcal{U}(t)$,
(iii) $[\![s]\!]_\rho > [\![t]\!]_\rho$ for some valuation ρ.
Then $\mathrm{NJ}(s, t)$.

Discrimination Pair

We now take term algebras for $\mathcal{F}$-algebras, and ordering on terms.

Definition. A pair $\langle \gtrsim, \succ \rangle$ of two relations $\gtrsim$ and $\succ$ is said to be a **discrimination pair** if (i) $\gtrsim$ is a rewrite relation, (ii) $\succ$ is a strict partial order and (iii) $\gtrsim \circ \succ \subseteq \succ$ and $\succ \circ \gtrsim \subseteq \succ$.

Theorem 3. Let $\mathcal{R}$ be a TRS and s, t terms. Suppose there exists a discrimination pair $\langle \gtrsim, \succ \rangle$ such that $\mathcal{U}(s) \subseteq \lesssim$, $\mathcal{U}(t) \subseteq \gtrsim$ and $s \succ t$. Then $\mathrm{NJ}(s, t)$.

(Proof Sketch) Since $\gtrsim$ is a rewrite relation, it follows that $u \to_{\{l \to r\}} v$ implies $u \lesssim v$ for any $l \to r \in \mathcal{U}(s)$, and $u \to_{\{l \to r\}} v$ implies $u \gtrsim v$ for any $l \to r \in \mathcal{U}(t)$.

Suppose $s \xrightarrow{*} u$ and $t \xrightarrow{*} u$. Let $s = s_0 \to s_1 \to \cdots \to s_n = u$. Then $s = s_0 \to_{\mathcal{U}(s)} s_1 \to_{\mathcal{U}(s)} \cdots \to_{\mathcal{U}(s)} s_n = u$. Thus $s \lesssim \cdots \lesssim u$. Since $t \prec s \lesssim \cdots \lesssim u$, we obtain $t \prec u$ by the property $\gtrsim \circ \succ \subseteq \succ$ of the discrimination pair.

Similarly, from $t \to \cdots \to u$, we obtain $t \gtrsim \cdots \gtrsim u$. By $u \succ t \gtrsim \cdots \gtrsim u$, we obtain $u \succ u$ by the property $\succ \circ \gtrsim \subseteq \succ$ of the discrimination pair.

This contradicts our assumption that $\succ$ is a strict partial order. $\square$

Argument Filtering for Non-Joinability

One can incorporates the same notion of argument filtering in dependency pairs.

An argument filtering is a mapping such that $\pi(f) \in \{[i_1, \ldots, i_k] \mid 1 \leq i_1 < \cdots < i_k \leq \mathrm{arity}(f)\} \cup \{i \mid 1 \leq i \leq \mathrm{arity}(f)\}$ for each $f \in \mathcal{F}$. We define $f(t_1, \ldots, t_n)^\pi = f(t_{i_1}^\pi, \ldots, t_{i_k}^\pi)$ if $\pi(f) = [i_1, \ldots, i_k]$, $f(t_1, \ldots, t_n)^\pi = t_i^\pi$ if $\pi(f) = i$. For TRS $\mathcal{R}$, we put $\mathcal{R}^\pi = \{l^\pi \to r^\pi \mid l \to r \in \mathcal{R}\}$.

Theorem 4. Let $\mathcal{R}$ be a TRS and s, t terms. Suppose there exists a discrimination pair $\langle \gtrsim, \succ \rangle$ and argument filtering π such that $\mathcal{U}_{\mathcal{R}^\pi}(s^\pi) \subseteq \lesssim$, $\mathcal{U}_{\mathcal{R}^\pi}(t^\pi) \subseteq \gtrsim$ and $s^\pi \succ t^\pi$. Then $\mathrm{NJ}(s, t)$.

Example 3.

$$\mathcal{R} = \left\{ \begin{array}{llll} (1) & \mathsf{c} \to \mathsf{f}(\mathsf{c},\mathsf{d}), & (3) & \mathsf{f}(x,y) \to \mathsf{h}(\mathsf{g}(y),x), \\ (2) & \mathsf{c} \to \mathsf{h}(\mathsf{c},\mathsf{d}) & (4) & \mathsf{h}(x,y) \to \mathsf{f}(\mathsf{g}(y),x) \end{array} \right\}.$$

Take candidates $\mathsf{h}(\mathsf{f}(\mathsf{c},\mathsf{d}),\mathsf{d})$ **and** $\mathsf{f}(\mathsf{c},\mathsf{d})$**.**

Take $\pi(\mathsf{g}) = 1$, $\pi(\mathsf{f}) = [2]$ **and** $\pi(\mathsf{h}) = [1]$**. Then** $\mathcal{U}(s^\pi) = \{(3)^\pi, (4)^\pi\}$ **and** $\mathcal{U}(t^\pi) = \{(3)^\pi, (4)^\pi\}$**.**

Then we obtain the constraint

$$\mathsf{h}(\mathsf{f}(\mathsf{d})) \succ \mathsf{f}(\mathsf{d}), \quad \mathsf{f}(y) \simeq \mathsf{h}(y), \quad \mathsf{h}(x) \simeq \mathsf{f}(x)$$

which is satisfied by a discrimination pair $\langle \gtrsim_{rpo}, \gtrsim_{rpo} \setminus \lesssim_{rpo} \rangle$ **with precedence** $\mathsf{f} \simeq \mathsf{h}$**. Thus** $\mathrm{NJ}(s,t)$**.**

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Implementation

We implemented our techniques on the confluence prover ACP.

- Interpretation by $\mathcal{F}$-algebras (Theorem 1) using the polynomial interpretation with linear polynomials and partition $\mathbb{N} = \biguplus_{0 \leq i < k} \{n \mid n \bmod k = i\}$ $(k = 2, 3)$.
- Interpretation by ordered $\mathcal{F}$-algebras (Theorem 2) with polynomial interpretation via linear polynomials.
- Descrimination pair (Theorem 4) using recursive path order with argument filtering.

Criteria are encoded as a constraint and an external SMT-solver is called to check it has a solution.

Experiments

	Th.1 ($k=2$)	Th.1 ($k=3$)	Th.2 (poly)	Th.4 (rpo)	all
Example 1	✓	✓	✓	✓	✓
Example 2	✓	✓	×	×	✓
Example 3	×	×	×	✓	✓
23 ex. (success/t.o.)	16/0	16/3	14/0	19/0	21/1
23 ex. (time)	25	293	206	26	84
35 ex. (success/t.o.)	17/5	16/8	17/3	17/1	16/9
35 ex. (time)	318	562	446	106	761

	ACP	CSI	Saigawa
Example 1	×	×	×
Example 2	×	×	×
Example 3	×	×	×
23 ex. (success/t.o.)	9/0	12/–	3/1
23 ex. (time)	2	2107	228
35 ex. (success/t.o.)	18/1	21/–	17/6
35 ex. (time)	71	485	482

23 new examples
35 examples from Cops
ACP v.0.31
CSI v.0.2
Saigawa v.1.4

Conclusion

Disproving confluence by showing non-joinability of candidates.

- **Proving non-joinability by interpretation**
  $\mathcal{F}$-algebra, usable rules
- **Proving non-joinability by ordering**
  ordered $\mathcal{F}$-algebra
  discrimination pairs, argument filtering
- **Implementation and experiments**

Future Works

- More effective interpretation and ordering