計算量クラス(前回の復習)

$\mathcal{P} \equiv \bigcup_{p:\text{多項式}} \mathrm{TIME}(p(l))$

$\mathcal{E} \equiv \bigcup_{c>1} \mathrm{TIME}(2^{cl})$

$\mathcal{EXP} \equiv \bigcup_{p:\text{多項式}} \mathrm{TIME}(2^{p(l)})$

(定義5.2) 集合Lがクラス$\mathcal{NP}$に入る⇔
以下を満たす多項式qと多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \exists w \in \Sigma^*: |w| \leqq q(|x|)[R(x,w)]$
略記: $\exists_q w \in \Sigma^*: [R(x,w)]$

(定理5.5) 集合Lがクラスco-$\mathcal{NP}$に入る⇔
以下を満たす多項式qと多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \forall w \in \Sigma^*: |w| \leqq q(|x|)[R(x,w)]$
略記: $\forall_q w \in \Sigma^*: [R(x,w)]$

Complexity Classes

$\mathcal{P} \equiv \bigcup_{p:\text{polynomial}} \mathrm{TIME}(p(l))$

$\mathcal{E} \equiv \bigcup_{c>1} \mathrm{TIME}(2^{cl})$

$\mathcal{EXP} \equiv \bigcup_{p:\text{polynomial}} \mathrm{TIME}(2^{p(l)})$

(Def 5.2) Set L is in the class $\mathcal{NP}$⇔
There exists a poly q and a poly-time computable pred. R s.t.
for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \exists w \in \Sigma^*: |w| \leqq q(|x|)[R(x,w)]$
Abbr. $\exists_q w \in \Sigma^*: [R(x,w)]$

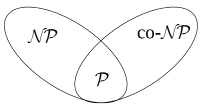
(Theorem 5.5) Set L is in the class co-$\mathcal{NP}$⇔
There exists a poly q and a poly-time computable pred. R s.t.
for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \forall w \in \Sigma^*: |w| \leqq q(|x|)[R(x,w)]$
Abbr. $\forall_q w \in \Sigma^*: [R(x,w)]$

定理5.8.
(1) $\mathcal{P} \subseteq \mathcal{NP}$, $\mathcal{P} \subseteq$ co-$\mathcal{NP}$ (よって, $\mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$)
(2) $\mathcal{NP} \subseteq \mathcal{EXP}$, co-$\mathcal{NP} \subseteq \mathcal{EXP}$ (よって, $\mathcal{NP} \cup$ co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

定理5.9.
(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP}$ → $\mathcal{NP}$ = co-$\mathcal{NP}$
(2) co-$\mathcal{NP} \subseteq \mathcal{NP}$ → $\mathcal{NP}$ = co-NP
(3) $\mathcal{NP} \neq$ co-$\mathcal{NP}$ → $\mathcal{P} \neq \mathcal{NP}$

補注:
(3)より, $\mathcal{NP} \neq$ co-$\mathcal{NP}$の証明は,
$\mathcal{P} \neq \mathcal{NP}$の証明より難しい.

$\mathcal{NP} \neq$ co-$\mathcal{NP}$が正しいと



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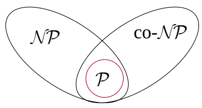
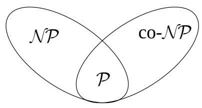


Theorem 5.8.
(1) $\mathcal{P} \subseteq \mathcal{NP}$, $\mathcal{P} \subseteq$ co-$\mathcal{NP}$ (thus, $\mathcal{P} \subseteq \mathcal{NP} \cap$ co-$\mathcal{NP}$)
(2) $\mathcal{NP} \subseteq \mathcal{EXP}$, co-$\mathcal{NP} \subseteq \mathcal{EXP}$ (thus, $\mathcal{NP} \cup$ co-$\mathcal{NP} \subseteq \mathcal{EXP}$)

Theorem 5.9.
(1) $\mathcal{NP} \subseteq$ co-$\mathcal{NP}$ → $\mathcal{NP}$ = co-$\mathcal{NP}$
(2) co-$\mathcal{NP} \subseteq \mathcal{NP}$ → $\mathcal{NP}$ = co-$\mathcal{NP}$
(3) $\mathcal{NP} \neq$ co-$\mathcal{NP}$ → $\mathcal{P} \neq \mathcal{NP}$

Note: from (3)
the proof for $\mathcal{NP} \neq$ co-$\mathcal{NP}$
is harder than that for $\mathcal{P} \neq \mathcal{NP}$.

If $\mathcal{NP} \neq$ co-$\mathcal{NP}$ is true,



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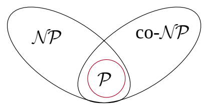

計算量クラス間の定義を概観すると…

クラス$\mathcal{P}$の定義(5章)
集合Lがクラス$\mathcal{P}$に入る ⇔
以下を満たす多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow R(x)$

クラス$\mathcal{NP}$の定義(定義5.2)
集合Lがクラス$\mathcal{NP}$に入る ⇔
以下を満たす多項式qと多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \exists w \in \Sigma^*: |w| \leqq q(|x|)[R(x,w)]$

クラスco-$\mathcal{NP}$の定義(定理5.5)
集合Lがクラスco-$\mathcal{NP}$に入る ⇔
以下を満たす多項式qと多項式時間計算可能述語Rが存在:
各 $x \in \Sigma^*$で$x \in L \Leftrightarrow \forall w \in \Sigma^*: |w| \leqq q(|x|)[R(x,w)]$

Observation of the definitions of the classes…

Def: Class $\mathcal{P}$ (Chapter 5)
Set L is in the class $\mathcal{P}$ ⇔
There exists a poly-time computable predicate R such that
for each $x \in \Sigma^*$, $x \in L \Leftrightarrow R(x)$

Def: Class $\mathcal{NP}$ (Def 5.2)
Set L is in the class $\mathcal{NP}$ ⇔
There exists a poly q and a poly-time computable pred. R s.t.
for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \exists w \in \Sigma^*: |w| \leqq q(|x|)[R(x,w)]$

Def: Class co-$\mathcal{NP}$ (Theorem 5.5)
Set L is in the class co-$\mathcal{NP}$ ⇔
There exists a poly q and a poly-time computable pred. R s.t.
for each $x \in \Sigma^*$, $x \in L \Leftrightarrow \forall w \in \Sigma^*: |w| \leqq q(|x|)[R(x,w)]$

$\exists x_1 \exists x_2 \exists x_3\, [R(x_1,x_2,x_3)] \Leftrightarrow \exists w(=<x_1,x_2,x_3>)[R'(w)]$

$\forall x_1 \forall x_2 \forall x_3\, [R(x_1,x_2,x_3)] \Leftrightarrow \forall w(=<x_1,x_2,x_3>)[R'(w)]$

…たとえば $\exists x \forall y \exists w[R(x,y,w)]$ は??

クラス Σ_k^p:$L = \{x : \exists_q w_1 \forall_q w_2 \ldots \Phi_q w_k [R(x, w_1, \ldots, w_k)]$

クラス Π_k^p:$L = \{x : \forall_q w_1 \exists_q w_2 \ldots \Phi_q w_k [R(x, w_1, \ldots, w_k)]$

(比較的)すぐわかる関係:

$\Sigma_0^p = \Pi_0^p = \mathcal{P}$ $\quad \Pi_k^p \subseteq \Pi_{k+1}^p \cap \Sigma_{k+1}^p$

$\Sigma_1^p = \mathcal{NP}$ $\quad \Sigma_k^p \subseteq \Pi_{k+1}^p \cap \Sigma_{k+1}^p$

$\Pi_1^p = \text{co-}\mathcal{NP}$

$\exists x_1 \exists x_2 \exists x_3\, [R(x_1,x_2,x_3)] \Leftrightarrow \exists w(=<x_1,x_2,x_3>)[R'(w)]$

$\forall x_1 \forall x_2 \forall x_3\, [R(x_1,x_2,x_3)] \Leftrightarrow \forall w(=<x_1,x_2,x_3>)[R'(w)]$

…How about, e.g., $\exists x \forall y \exists w[R(x,y,w)]$??

Class $\Sigma_k^p : L = \{x : \exists w_1 \forall w_2 \ldots \Phi\, w_k [R(x, w_1, \ldots, w_k)]$

Class $\Pi_k^p : L = \{x : \forall w_1 \exists w_2 \ldots \Phi\, w_k [R(x, w_1, \ldots, w_k)]$

It is not difficult to see that…

$\Sigma_0^p = \Pi_0^p = \mathcal{P}$ $\quad \Pi_k^p \subseteq \Pi_{k+1}^p \cap \Sigma_{k+1}^p$

$\Sigma_1^p = \mathcal{NP}$ $\quad \Sigma_k^p \subseteq \Pi_{k+1}^p \cap \Sigma_{k+1}^p$

$\Pi_1^p = \text{co-}\mathcal{NP}$

(比較的)すぐわかる関係:

$\Sigma_0^p = \Pi_0^p = \mathcal{P}$; $\Sigma_1^p = \mathcal{NP} \to \Sigma_2^p \to \Sigma_3^p \to \Sigma_4^p$; $\Pi_1^p = \text{co-}\mathcal{NP} \to \Pi_2^p \to \Pi_3^p \to \Pi_4^p$; $\mathcal{EXP}$

$$\mathcal{PH} \equiv \bigcup_k \Sigma_k^p = \bigcup_k \Pi_k^p$$

戸田の定理(1991): $\mathcal{PH} \subseteq \mathcal{P}^{\mathcal{PP}}$

祝!!
ゲーデル賞
(1998)

It is not difficult to see that…

$\Sigma_0^p = \Pi_0^p = \mathcal{P}$; $\Sigma_1^p = \mathcal{NP} \to \Sigma_2^p \to \Sigma_3^p \to \Sigma_4^p$; $\Pi_1^p = \text{co-}\mathcal{NP} \to \Pi_2^p \to \Pi_3^p \to \Pi_4^p$; $\mathcal{EXP}$

$$\mathcal{PH} \equiv \bigcup_k \Sigma_k^p = \bigcup_k \Pi_k^p$$

Toda's Theorem(1991): $\mathcal{PH} \subseteq \mathcal{P}^{\mathcal{PP}}$

Gödel
Prize!!
(1998)