I618 Advanced Computer Science II (Part II)

12/21 11:00-12:30
1/ 7 15:10-16:40
1/ 9 9:20-10:50
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Introduction

- Representative approaches to ($\mathcal{NP}$-)hard problems are…
  - approximation algorithms
  - exact algorithms with exponential time
  - <u>restrictions on inputs</u>
    - <u>some special graph classes</u>

Introduction

A graph $G=(V,E)$ is an *intersection graph* over set V of objects iff $\{v,u\}$ is in E if and only if corresponding objects are overlapping.

- We will mainly discuss about
  - Chordal graphs and interval graphs
    - typical intersection graphs
    - many applications
      - matrix manipulation, bioinformatics, scheduling, …
    - many useful graph theoretic properties
      - typical subclasses of Perfect Graphs

1960 [Berge]: Strong Perfect Graph Conjecture

↓

2002 [Chudnovsky, Cojnuejols, Liu, Seymour, and Vuskovic]:
Strong Perfect Graph Theorem

Introduction

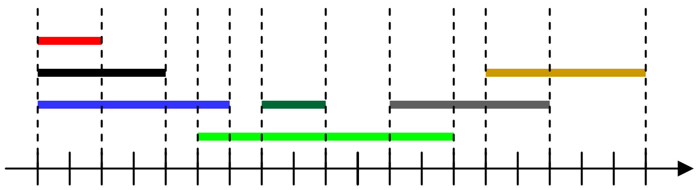
- We will mainly discuss about
  - Chordal graphs and interval graphs
    - typical intersection graphs
    - <u>many applications</u>
      - Matrix manipulation, bioinformatics, scheduling, …
    - <u>many useful graph theoretic properties</u>
      - typical subclasses of Perfect Graphs
  - ○ many $\mathcal{NP}$-hard problems become *tractable* on those graph classes
  - △ several problems are *still hard* on those graph classes

Interval Graphs

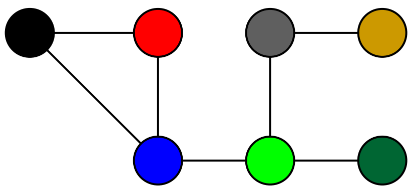
[Today's Goal]
For any given interval graph, its maximum clique can be found in linear time.
(C.f., the maximum clique problem is $\mathcal{NP}$-complete in general.)

- Simplest intersection graphs
  - Since 195?- (Hajos (Graph theorist) & Benzer (Biologist))

[Definition 1] A graph $G=(V,E)$ with $V=\{v_1,v_2,\ldots,v_n\}$ is an *interval graph* if and only if there is a set $\mathcal{I}$ of intervals $\{I_1, I_2,\ldots, I_n\}$ such that $\{v_i, v_j\}\in E$ if and only if I_i intersects I_j. We call $\mathcal{I}$ an *interval representation* of G.



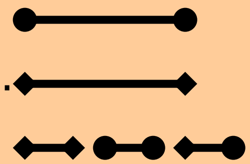
interval representation



corresponding graph

Interval Graphs

[Description]
open interval…
closed interval…
mixed interval…



- Interval representations of an interval graph
  - is an interval open or closed?
    - *open*… e.g., (1,5) does *not* contain the value 5.
    - *closed*… e.g., [2,8] contains the value 8.
    - Let $\mathcal{C}_o$, $\mathcal{C}_c$, $\mathcal{C}_m$ be the classes of interval graphs that consist of open intervals, closed intervals, and mixed, respectively.

[Theorem 1] $\mathcal{C}_o=\mathcal{C}_c=\mathcal{C}_m$

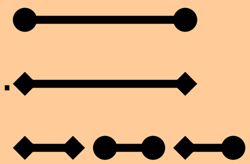
[Notation]
For an interval I, we denote the left endpoint by $L(I)$, and the right endpoint by $R(I)$.

(Proof) We show that ① $\mathcal{C}_o \subseteq \mathcal{C}_c$ ② $\mathcal{C}_c \subseteq \mathcal{C}_m$ and ③ $\mathcal{C}_m \subseteq \mathcal{C}_o$.

① Let $\mathcal{I}_o$ be an interval representation of an interval graph G such that $\mathcal{I}_o$ only contains <u>open</u> intervals. Then, we construct $\mathcal{I}_c$ that is an interval representation of G and $\mathcal{I}_c$ only contains <u>closed</u> intervals as follows.

Interval Graphs

[Description]
open interval…
closed interval…
mixed interval…



- Interval representations of an interval graph

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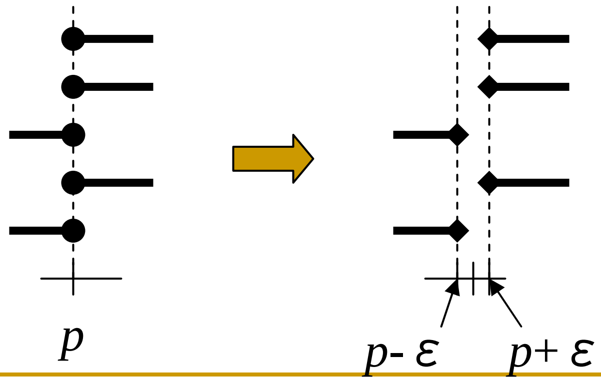
For each point p that is an endpoint of at least one interval, we modify the intervals as follows for sufficiently small ε :

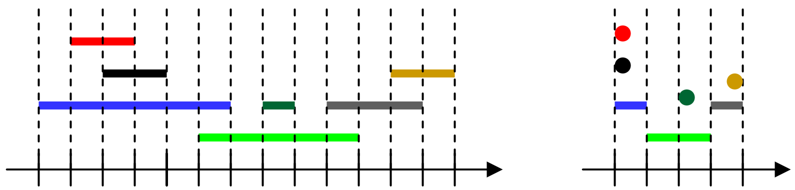


Repeating this process, we can obtain a closed interval representation $\mathcal{I}_c$ of G.

② is trivial, and ③ is similar to ①.
Hence we have the theorem. □

Interval Graphs

- Interval representations of an interval graph
  - Hereafter,
    - By Theorem 1, we assume that all intervals are closed.
    - All endpoints are integers, and leftmost endpoint is 0.
  - We have two natural interval models;
    1. Each endpoint takes distinct value in $[0..2n\text{-}1]$ with n vertices (conversely, each integer in $[0..2n\text{-}1]$ corresponds to exactly one endpoint).
    2. We admit $L(I)=R(I)$, that is, the length of an interval can be 0, and intervals have *no redundancy*.



We call the second type "*compact representation*".

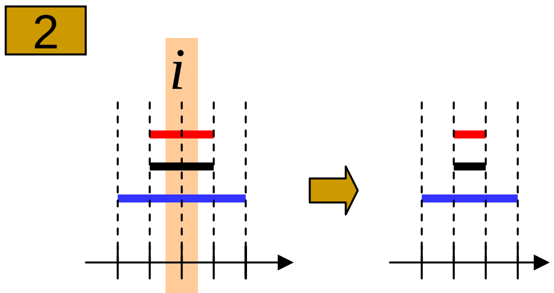
Interval Graphs

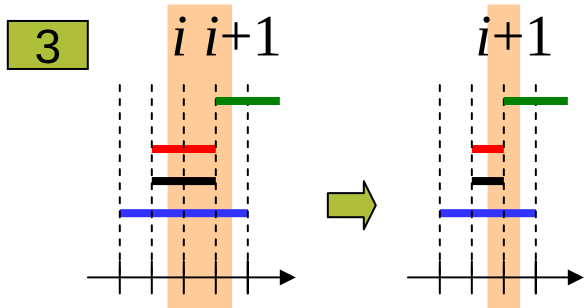
[Notation]
For a point p, let $N[p]$ denote the set of intervals that contain p.

- *Compact* interval representations of an interval graph

[Definition 2] An interval representation $\mathcal{I}$ is called *compact* if it satisfies the following conditions;

1. (all endpoints are integers and the leftmost endpoint is 0,)
2. each integer i corresponds to at least one endpoint with $0 \leqq i \leqq k$ for some positive integer k, and
3. for each integer i with $0 \leqq i < k$, we have $N[i] \not\subset N[i+1]$ and $N[i+1] \not\subset N[i]$.





Interval Graphs

[Notation]
For an interval representation $\mathcal{I}$,
we denote by $L(\mathcal{I}) := \min_{I\in\mathcal{I}} L(I)$
and by $R(\mathcal{I}) := \max_{I\in\mathcal{I}} R(I)$

- *Compact* interval representations of an interval graph

[Theorem 2] Let $\mathcal{I}$ be a compact interval representation of a *connected* interval graph $G=(V,E)$ of n vertices with $n \geqq 2$. Then $L(\mathcal{I})=0$ and $R(\mathcal{I})=k$ for some integer k. Then, $k \leqq n-2$.

[Lemma 1] Let $\mathcal{I}$ be a compact interval representation of a *connected* interval graph $G=(V,E)$. Then there exists an interval $I \in \mathcal{I}$ such that $[L(I),R(I)]=[0,0]$.

(Proof) of Lemma 1. We have two cases;

1. $[L(\mathcal{I}),R(\mathcal{I})]=[0,0]$ (C.f. G is a complete graph): Trivial.
2. $R(\mathcal{I})>0$: If there are no such intervals, we have $N[1]=N[2]$ or $N[1]\subset N[2]$. Both cases contradict to the assumption that $\mathcal{I}$ is a compact interval representation. □

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(Proof) of Theorem 2. We prove by induction for k.

1. $k=0$: The graph G is a complete graph, and easy to see that $k \leqq n-2$.
2. $k>0$: By Lemma 1, there are x intervals I with $R[I]=L[I]=0$ with $x>0$. We then remove them from $\mathcal{I}$ and obtain $\mathcal{I}$' with $n-x$ intervals. Then, by the inductive hypothesis, we have $k-1 \leqq n-x-2$. Hence we have $k \leqq n-2$ since $x>0$. □

Interval Graphs

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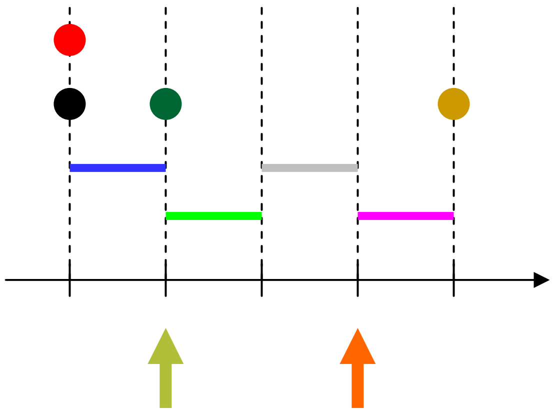
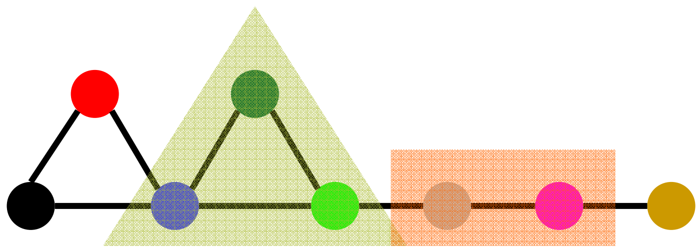
- *Compact* interval representations of an interval graph

[Theorem 3] Let $\mathcal{I}$ be a compact interval representation of a *connected* interval graph $G=(V,E)$ of n vertices with $n \geqq 2$. Then $N[i]$ induces a maximal clique of G for each i in $[L(\mathcal{I}),R(\mathcal{I})]$. Moreover, each maximal clique M of G satisfies $M=N[i]$ for some i. That is, they make one-to-one mapping.





Interval Graphs

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- *Compact* interval representations of an interval graph

(Proof) of latter half which says a maximal clique M satisfies $M=N[i]$ for some i.

To derive a contradiction, we assume that there are no such index i. Let i' be the index such that $|N(i') \cap M| \geqq |N(i'') \cap M|$ for any other i''. Then there is an interval I_j such that $v_j \in M$ and $I_j \notin N[i]$. Without loss of generality, we assume that $R(I_j)<i$.

By assumption of i', there is a vertex $v_k \in M$ such that $I_k \in N(i')$ and $I_k \notin N[R(I_j)]$ since $|N(i') \cap M| \geqq |N(R(I_j)) \cap M|$ and $I_j \in N(R(I_j))-N(i')$. Then, I_k and I_j cannot intersect, which contradicts that M contains v_k and v_j.

Interval Graphs

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- *Compact* interval representations of an interval graph

(Proof) of former half which says $N[i]$ induces a maximal clique M for each i.

It is easy to see that $N[i]$ induces a clique C. Hence we show C is maximal. To derive a contradiction, we assume that $C \subset M$ for some maximal clique M.

Then, by the latter half of the proof, there exists j such that $N[j]$ induces M. Without loss of generality, we assume $i<j$.

Then, it is not difficult to see that there are two indices i' and j' with $i \leqq i' < j' \leqq j$ such that $N[i'] \subset N[j']$, which contradicts that $\mathcal{I}$ is compact. □

Interval Graphs

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- *Compact* interval representations of an interval graph

[Theorem 4] Any connected interval graph $G=(V,E)$ with $|V|>1$ has at most $|V|-1$ maximal cliques.

(Proof) Immediately from Theorems 2 & 3. □

[Theorem 5] For any connected interval graph $G=(V,E)$ given in a compact interval representation form, its maximum clique can be found in $O(|V|)$ time.

(Proof) Roughly, sweep the interval representation and check $N[i]$ for each integer i. Details will be discussed in the future class with suitable data structure. □