Simple Undecidable Problem on Origami

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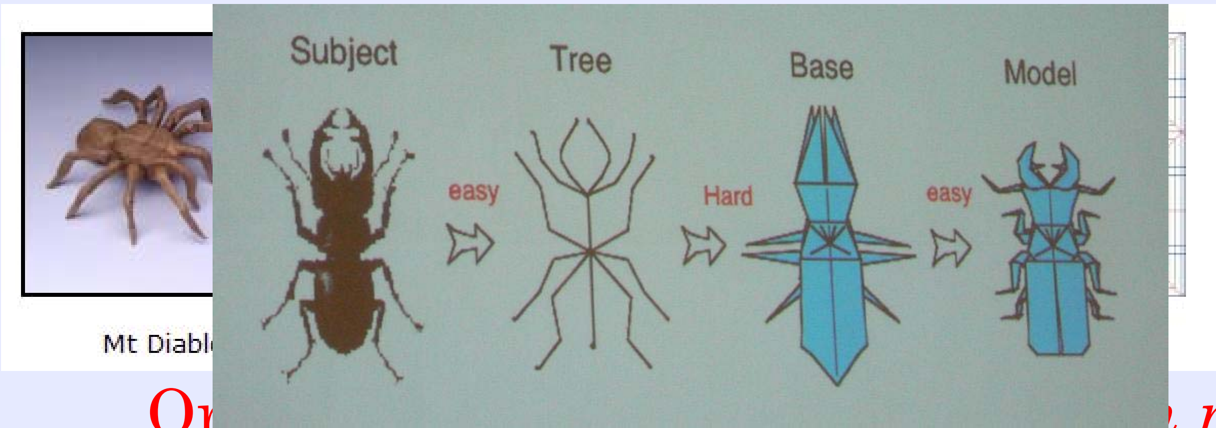
Computational Origami

- Intractable results:
  - The complexity of Flat Origami
    Bern and Hayes, *SODA*, 1996.
- Tractable results:
  - *TreeMaker*; Free software by R. Lang
    given a metric tree, it generates the development.



Uehara:
- NP-hardness of a Pop-up book (2006)
- Efficient algorithms for pleat folding (2010)

Or *model*”?

Complexity/Efficiency on Origami(?)

- From the viewpoint of Theoretical Computer Science…
- E.g., *Two Resources* on Turing Machine Model
  1. Time: The number of applied operations
  2. Space: The number of memory cells required to compute

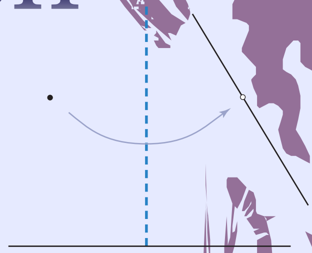
Complexity / Effici
Origam

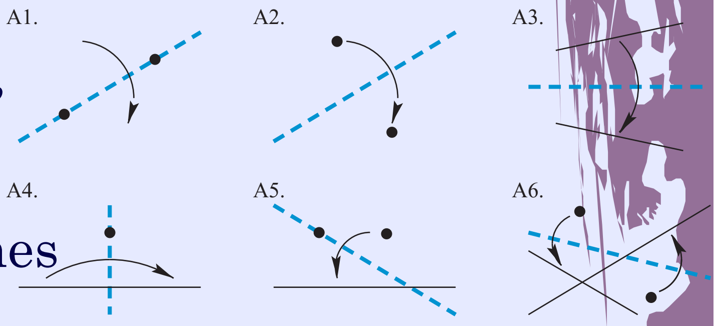
Wait a moment!
At first, what is the
"computation model"
corresponding to Turing
Machine?

- From the viewpo
  Computer Scienc
- *Two Resources* on ORIGAM
  1. Time…The number of fold ng(basic operation)
     - J. Cardinal, *E. D. Demaine*, *M. L. Demaine*, S. Imahori, T. Ito, M. Kiyomi, *S. Langerman*, R. Uehara, and T. Uno: Algorithmic Folding Complexity, *Graphs and Combinatorics*, Vol. 27, pp. 341-351, 2011.
  2. Space…???
     - R. Uehara: Stretch Minimization Problem of a Strip Paper, *5th International Conference on Origami in Science, Mathematics and Education*, 2010/7/13-17.
     - R. Uehara: On Stretch Minimization Problem on Unit Strip Paper, *22nd Canadian Conference on Computational Geometry*, pp. 223-226, 2010/8/9-11.

Origami as a computation model？

- Origami as a “computation model”
  - Input：“points” on a sheet of square paper
  - Basic operations：
    - 7 operations by “Huzita & Hatori”
  - Comparison & branch：
    - decision of coincidence of points/lines





- finite operations of “straight edge and compass”
  - can solve quadratic equations
- finite combinations of 7 basic operations above
  - can solve quartic equations
  - (E.g., can trisect any angle)

…They do *not* deal with “computability” and/or “computational complexity” of an Origami

Origami as a computation model?

- "Reasonable" Origami model would be...
  - Given: finite number of points on a sheet of paper
  - Operation: 7 basic operations proposed by Huzita and Hatori
  - Each point has a coordinate (x,y) with real numbers x and y
  - "a point" and "a line";
    - We can "*use*" it (if it exists) to make another one
    - We can compare accuracy the coincidence between two "points" which can be an intersect of two or more lines
    - "Nonexistent point/line" (which may be goal) can be "*seen*", but cannot be "*used*"

Origami as a computation model?

- "Reasonable" Origami model would be…
  - Given: finite number of points on a sheet of paper
  - Operation: 7 basic operations proposed by Huzita and Hatori
  - Each point has a coordinate (x,y) with real numbers x and y

[Key points]

- Points on an origami have coordinates (x,y), which are real numbers. Thus, they are uncountable infinity.
- Sequence of operations are countable infinity.

Big Gap!!

⇒Natural "undecidable" problem…

Undecidable problem on Origami

- Consider the following simple (?) **foldability** problem:

  Input: Three "start points" (x, y, z) and a "goal point" w on a unit square paper

  Question: Folding from points (x, y, z), after finite number of foldings, can you make two lines l_1, l_2 such that their intersection coincides to w?

- Simpler **foldability** on 1D Origami:

  Input: Three "start points" (x, y, z) and a "goal point" w on a line segment [0,1]

  Question: Folding from points (x, y, z), after finite number of foldings, can you fold at w?

[Theorem]

> Foldability is undecidable even on 1D Origami

That is, we cannot make a program that always answers either [Yes] or [No].

Undecidable problem on Origami

[Theorem]

Foldability is undecidable even on 1D Origami

[Outline of the proof]

To derive a contradiction, we assume that a program (or some algorithmic way) P solves it. Then, for fixed x,y,z, we define point sets $\mathrm{S}i$ according to the step i of $\mathrm{P}(x,y,z,w)$;

$$\mathrm{S}i = \{\, w \mid \mathrm{P}(x,y,z,w) \text{ halts after the } i\text{th step for } w \}$$

Then, $|\mathrm{S}i|$ is countable, and so is $\cup \mathrm{S}i$.

By a diagonalization, we can construct w such that $\mathrm{P}(x,y,z,w)$ never halt in a finite step. □

Not so trivial.

Undecidable problem on Origami

[Theorem]

Foldability is undecidable even on 1D Origami

[Yes/No]

[Outline of the proof (cont.)]

Si = { w | P(x,y,z,w) halts after the ith step for w}

- "Yes": "points coincide with the other existing points"
  ⇒ countable!
- <u>"No" : may be for uncountable many *w*?</u>

⇒ "No" to all real numbers in (a,b)

- We can make a point p in (a,b) with finite operations; hence p in (a,b) is a "Yes" instance, a contradiction.

∴ "No" points are also countable, and |Si| is countable.

So what? …what this theorem means

- Undecidability of origami…
  - The halting problem on TM implies a kind of "strongness" of the machine model.
  - So it implies "strongness" of an origami model in a paradoxical way?

- Future works…
  - Model admitting error ε:
    Ex: "real number *r*" is represented by [*r*-ε, *r*+ε]
  - From the viewpoint of algorithms:
    Ex: "Polynomial time constructible real numbers" by Origami?

Thank You!