I216 - Answers and Comments on Report

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May 02, 2017

Problem 1.1

Problem 1.1

Let $X_1, X_2, \ldots$ be the Turing machines, and $x_1, x_2, \ldots$ are their corresponding binary string. (That is, a string x_i is the binary code of the Turing machine X_i.) We denote the output of X_i with a binary input x by $X_i(x)$. For two strings x and y, their concatenation is denoted by $x \cdot y$ (e.g., $000 \cdot 111 = 000111$). Let f be the function defined as follows:

$$f(x) = \begin{cases} X_i(x_i) \cdot X_i(x_i) & \text{if } X_i \text{ halts for the input } x = x_i \\ 0 & \text{otherwise} \end{cases}$$

Prove that this function f is not computable.

Problem 1.1 (Answer)

Let F be the set of computable functions (= Turing machines). Since F is countable, we can list elements of F as $X_1, X_2, \dots$. We define the below table as follows: if $X_i(x_j)$ halts, put the value of $X_i(x_j)$ to position (i, j); otherwise, put $\perp$.

	x_1	x_2	x_3	$\dots$	x_i	$\dots$
X_1	00_{0000}	01	11	$\dots$	$\perp$	$\dots$
X_2	1	$\perp_0$	11	$\dots$	1	$\dots$
X_3	0	1	1_{11}	$\dots$	0	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
X_i	0	00	10	$\dots$	01_{0101}	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

Suppose that the given function $f(x)$ is computable. Then $f(x)$ is computed by a Turing machine X_j for some $j \in \{1, 2, \dots\}$. But now, consider the value of $X_j(x_j)$

$$X_j(x_j) = f(x_j) = \begin{cases} X_j(x_j) \cdot X_j(x_j) & \text{if } X_j \text{ halts for the input } x = x_j \\ 0 & \text{otherwise} \end{cases}$$

Hence, the value of $X_j(x_j)$ in the above table cannot be defined. It follows that $f(x)$ is not computable.

Problem 1.2

Problem 1.2

The set $\mathbb{N}$ of natural numbers is countable. Now, prove that the set $2^{\mathbb{N}}$ of subsets of $\mathbb{N}$ is *not* countable by diagonalization. (Hint: For $S = \{1, 2, 3\}$ we have $2^S = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.)

Problem 1.2 (Answer)

Suppose that the set $2^{\mathbb{N}}$ is countable. Hence, we can list elements of $2^{\mathbb{N}}$ as $N_0, N_1, N_2, \ldots$, where each N_i is a subset of $\mathbb{N}$ for some $i \in \mathbb{N}$. Next, we define the below table as follows: for $j \in \mathbb{N}$, put 1 to position (i, j) if $j \in N_i$; otherwise, put 0.

	0	1	2	...	i	...
N_0	1_0	0	1	...	1	...
N_1	0	1_0	0	...	0	...
N_2	0	1	0_1	...	1	...
...	...	...	...	...	...	...
N_i	1	0	1	...	0_1	...
...	...	...	...	...	...	...

Let $A = \{i \mid i \in \mathbb{N} \text{ and } i \notin N_i\}$. In the above table, 1 means $i \in A$ and 0 means $i \notin A$, where $i \in \mathbb{N}$.
Then, A is a subset of $\mathbb{N}$. It follows that $A = N_j$ for some $j \in \mathbb{N}$. But now, we have $j \in A$ if and only if $j \notin N_j = A$. Thus, the value at position (j, j) of the above table cannot be decided. Therefore, $2^{\mathbb{N}}$ is not countable.

Problem 1.3

Problem 1.3

In the slide of the second lecture, we prove the theorem that claims “The set R of real numbers is not countable.” Now let replace every “real” by “rational”. Then it seems that we prove the theorem that claims “The set R' of rational numbers is not countable.” But, the set of all rational numbers is countable. Point out where is wrong.

Problem 1.3 (Answer)

4. Undecidability and Diagonalization

4. 2. Diagonalization

Theorem:

The set R of real numbers is *not* countable.

[Proof by *diagonalization*]

Assume that P is countable; i.e., they are enumerated as $R = \{ R_0, R_1, R_2, R_3, \ldots \}$

Each R_i is in the form of $R_i = \ldots r_{i,4}' \, r_{i,3}' \, r_{i,2}' \, r_{i,1}' \, r_{i,0} \,.\, r_{i,1} \, r_{i,2} \, r_{i,3} \, r_{i,4} \ldots$ in decimal.

We define a number $X = 0.\, x_1 \, x_2 \, x_3 \ldots$ by

$$\begin{cases} x_i = 3 \text{ if } r_{i,i} = 1,2,4,5,6,7,8,9, \text{ or } 0 \\ x_i = 1 \text{ if } r_{i,i} = 3 \end{cases}$$

Then X is a real number, so it will appear as $X = R_i$ for some i.

But x_i is... 3? or 1?... we cannot decide it, which is a contradiction!

Therefore P is not countable!!

Ex.

$R_0 = 123.\underline{4}56\ldots$

$R_1 = 0.1\underline{3}1313\ldots$

$R_2 = 555.55\underline{5}5555\ldots$

$R_3 = 3.141\underline{5}92\ldots$

...

$X = 0.\,3133\ldots$

The original slide.

Problem 1.3 (Answer)

4. Undecidability and Diagonalization

4. 2. Diagonalization

Theorem:

The set R of ~~real~~ rational numbers is *not* countable.

[Proof by *diagonalization*]

Assume that P is countable; i.e., they are enumerated as $R = \{ R_0, R_1, R_2, R_3, \dots \}$

Each R_i is in the form of $R_i = \dots r_{i,4}' \, r_{i,3}' \, r_{i,2}' \, r_{i,1}' \, r_{i,0} \, . \, r_{i,1} \, r_{i,2} \, r_{i,3} \, r_{i,4} \dots$ in decimal.

We define a number $X = 0. \, x_1 \, x_2 \, x_3 \dots$ by

$$\begin{cases} x_i = 3 \text{ if } r_{i,i} = 1,2,4,5,6,7,8,9, \text{ or } 0 \\ x_i = 1 \text{ if } r_{i,i} = 3 \end{cases}$$

Then X is a ~~real~~ rational number, so it will appear as $X=R_i$ for some i.

But x_i is... 3? or 1?... we cannot decide it,
which is a contradiction!

Therefore P is not countable!!

Ex.
R_0 = 123.456...
R_1 = 0.131313...
R_2 = 555.5555555...
R_3 = 3.141592...
...

X = 0. 3133...

Replacing "real" by "rational".

Problem 1.3 (Answer)

4. Undecidability and Diagonalization

4. 2. Diagonalization

Theorem:

The set R of ~~real~~ rational numbers is *not* countable.

[Proof by *diagonalization*]

Assume that P is countable; i.e., they are enumerated as $R = \{ R_0, R_1, R_2, R_3, \ldots \}$

Each R_i is in the form of $R_i = \ldots r_{i,4}' \, r_{i,3}' \, r_{i,2}' \, r_{i,1}' \, r_{i,0} \, . \, r_{i,1} \, r_{i,2} \, r_{i,3} \, r_{i,4} \ldots$ in decimal.

We define a number $X = 0. \, x_1 \, x_2 \, x_3 \ldots$ by

$$\begin{cases} x_i = 3 \text{ if } r_{i,i} = 1,2,4,5,6,7,8,9, \text{ or } 0 \\ x_i = 1 \text{ if } r_{i,i} = 3 \end{cases}$$

Then X is a ~~real~~ rational number, so it will appear as $X=R_i$ for some i. ← Must be wrong here!

But x_i is… 3? or 1?… we cannot decide it,
which is a contradiction!
Therefore P is not countable!!

Ex.
R_0 = 123.<u>4</u>56…
R_1 = 0.1<u>3</u>1313…
R_2 = 555.55<u>5</u>5555…
R_3 = 3.141<u>5</u>92…
…
X = 0. 3133…

Replacing "real" by "rational".

Problem 2.1

Problem 2.1

Determine if each of the following equations is correct or wrong. If it is correct, prove it. If it is wrong, disprove it. You can use l'Hospital's rule if you need it.

(1) $3n^3 + 4n^2 = O(n^2 + n)$

(2) $3n^2 + 3n = O(n^8 + 2)$

(3) $n = O(\log n)$

(4) $n^8 = O(2^n)$

l'Hospital's rule

For functions f and g which are differentiable on an open interval I except possibly at a point c contained in I, if

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0 \text{ or } \pm\infty, g'(x) \neq 0 \text{ for all } x \in I \text{ with } x \neq c$$
$$\text{and } \lim_{x \to c} \frac{f'(x)}{g'(x)} \text{ exists, then} \lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Problem 2.1 (Answer)

(1) $3n^3 + 4n^2 = O(n^2 + n)$. [False]
Suppose to the contrary that there exist positive constants c and n_0 such that for every $n \geq n_0$, we have $3n^3 + 4n^2 \leq c(n^2 + n)$. For $n > cn_0 \geq n_0$, we have

$$3n^3 + 4n^2 \geq 3n(n^2 + n) > 3cn_0(n^2 + n) \geq c(n^2 + n), \tag{1}$$

which is a contradiction.

(2) $3n^2 + 3n = O(n^8 + 2)$. [True]
For every n, we have

$$3n^2 + 3n \leq 3n^2 + 3n^2 = 6n^2 \leq 6n^8 < 6(n^8 + 2). \tag{2}$$

Choose $c = 6$ and $n_0 = 1$ then for every $n \geq n_0$, $3n^2 + 3n \leq c(n^8 + 2)$.

Problem 2.2 (Answer)

Lemma

$$f(n) = O(g(n)) \Leftrightarrow \lim_{n\to\infty} \frac{f(n)}{g(n)} \in [0, \infty).$$

Without loss of generality, assume $\log n$ is the **natural logarithm** of n.

(3) $n = O(\log n)$. [False]
Use the Lemma above and l'Hospital's rule, we have

$$\lim_{n\to\infty} \frac{n}{\log n} \overset{\text{l'Hospital}}{=} \lim_{n\to\infty} \frac{1}{1/n} = \infty \notin [0, \infty). \tag{3}$$

(4) $n^8 = O(2^n)$. [True]
Use the Lemma above and l'Hospital's rule, we have

$$\lim_{n\to\infty} \frac{n^8}{2^n} \overset{\text{l'Hospital}}{=} \lim_{n\to\infty} \frac{8n^7}{2^n \log 2} \overset{\text{l'Hospital}}{=} \dots \overset{\text{l'Hospital}}{=} \lim_{n\to\infty} \frac{8!}{2^n (\log 2)^8} = 0. \tag{4}$$

Problem 2.2

Problem 2.2

In the definition of time complexity, we estimate running time under the assumption of "the worst case." If we can assume that an input is given uniformly at random, define "the average case time complexity" under the assumption.

Problem 2.2 (Answer)

Let $\mathcal{I}$ be the set of inputs of a given problem X. Suppose we have an algorithm A which solves X. Then, the asymptotic "worst case" time complexity of A with inputs of size n, denoted by $T(n)$, is defined as follows.

$$T(n) = \max_{x \in \mathcal{I}, |x| = n} T(x), \tag{5}$$

where $T(x)$ is the time A solves X with input $x \in \mathcal{I}$.
Now, assume that an input is given uniformly at random, i.e., each input $x \in \mathcal{I}$ is given to the algorithm with probability $\mathsf{Pr}(x) = \frac{1}{|\mathcal{I}|}$. Then, "the average case time complexity" of A with inputs of size n, denoted by $T_{\text{avg}}(n)$, can be defined as follows.

$$T_{\text{avg}}(n) = \sum_{x \in \mathcal{I}, |x| = n} \mathsf{Pr}(x) T(x) = \frac{\sum_{x \in \mathcal{I}, |x| = n} T(x)}{|\mathcal{I}|}. \tag{6}$$