実践的幾何アルゴリズム
Advanced Algorithms for Computational Geometry

5. 計算幾何の基礎の基礎

担当：上原隆平

Advanced Algorithms for Computational Geometry
実践的幾何アルゴリズム

5. Introduction to Computational Geometry

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計算幾何学の基礎(1)

点の表現:

- 座標を用いて表現するのが最も一般的
- 2次元の場合には，p(x,y) のように表現

典型的な2点間の「距離」

- ユークリッド距離: $\mathrm{dist}(p_1,p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- マンハッタン距離: $\mathrm{dist}(p_1,p_2) = |x_1 - x_2| + |y_1 - y_2|$
- L_p 距離: $\mathrm{dist}(p_1,p_2) = \sqrt[p]{(x_1 - x_2)^p + (y_1 - y_2)^p}$
- L_∞ 距離: $\mathrm{dist}(p_1,p_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

Introduction to Computational Geometry (1)

Representation of points:

- We use the standard coordinate system
- In 2D, we represent as p(x,y)

Representative "Distances" between two points;

- Euclidean distance: dist$(p_1,p_2) = \sqrt{(x_1 - x_2)^2+(y_1 - y_2)^2}$
- Manhattan distance: dist$(p_1,p_2) = |x_1 - x_2| + |y_1 - y_2|$
- L_p distance: dist$(p_1,p_2) = \sqrt[p]{(x_1 - x_2)^p+(y_1 - y_2)^p}$
- L_∞ distance: dist$(p_1,p_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

計算幾何学の基礎(2)

直線の表現:

1. 傾きとy切片による表現: $y = ax + b$
   - y軸に平行な直線($x = c$)は表現できない…
2. 3つのパラメタを用いた表現: $ax + by + c = 0$
   - 与えられた直線に対して，表現が一意的に定まらない…
3. 2点を通る直線としての表現:
   - 2点$(p, q), (r, s)$を通る直線; $(s - q)x - (r - p)y = ps - qr$

3番目の表現における2直線の交点;

与えられた2直線

$$l_1 : (y_2 - y_1)x - (x_2 - x_1)y = x_1 y_2 - x_2 y_1$$
$$l_2 : (y_4 - y_3)x - (x_4 - x_3)y = x_3 y_4 - x_4 y_3$$

に対してその交点 (x,y) は

$$x = \frac{\xi(x_4 - x_3) - \eta(x_2 - x_1)}{\Delta}, \quad y = \frac{\xi(y_4 - y_3) - \eta(y_2 - y_1)}{\Delta}$$

$$\xi = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \eta = \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}, \Delta = \begin{vmatrix} y_2 - y_1 & x_2 - x_1 \\ y_4 - y_3 & x_4 - x_3 \end{vmatrix}$$

Introduction to Computational Geometry (2)

Representations of lines:

1. Representation by slope and y-intercept: $y = ax + b$
   - The lines in parallel to y-axis (in the form of $x = c$) cannot be represented…
2. Representation by three parameters: $ax + by + c = 0$
   - For a given line, its representation is not determined uniquely…
3. Representation by two points:
   - For two points $(p, q), (r, s)$; $(s - q)x - (r - p)y = ps - qr$

In the third representation, the intersect of two lines is;

For two lines

$$l_1 : (y_2 - y_1)x - (x_2 - x_1)y = x_1 y_2 - x_2 y_1$$
$$l_2 : (y_4 - y_3)x - (x_4 - x_3)y = x_3 y_4 - x_4 y_3$$

There intersection (x,y) is

$$x = \frac{\xi(x_4 - x_3) - \eta(x_2 - x_1)}{\Delta}, y = \frac{\xi(y_4 - y_3) - \eta(y_2 - y_1)}{\Delta}$$

$$\xi = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}, \eta = \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix}, \Delta = \begin{vmatrix} y_2 - y_1 & x_2 - x_1 \\ y_4 - y_3 & x_4 - x_3 \end{vmatrix}$$

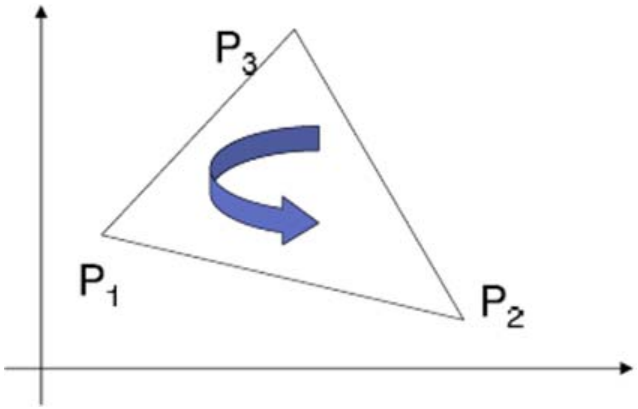
計算幾何学の基礎(3)

線分のパラメータ表示

2点$(x_1,y_1),(x_2,y_2)$を結ぶ線分:$\mu \in [0,1]$ $\begin{cases} x = (1-\mu)x_1+\mu x_2 \\ y = (1-\mu)y_1+\mu y_2 \end{cases}$

三角形の面積

面積= $\frac{1}{2}((x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1))$



面積>0: 反時計回り
面積<0: 時計回り
面積=0: 一直線上の3点

三角形の符号付面積

Introduction to Computational Geometry (3)

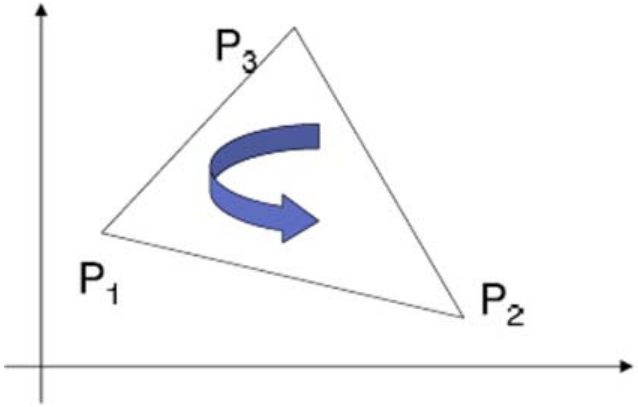
Representations of a line with parameter

Line segment joining two points $(x_1,y_1),(x_2,y_2):\mu \in [0,1]$

$$\begin{cases} x = (1-\mu)x_1+\mu x_2 \\ y = (1-\mu)y_1+\mu y_2 \end{cases}$$

Surface area of a triangle

Area= $\frac{1}{2}((x_2-x_1)(y_3-y_1)-(x_3-x_1)(y_2-y_1))$



Area>0: Counterclockwise

Area<0: Clockwise

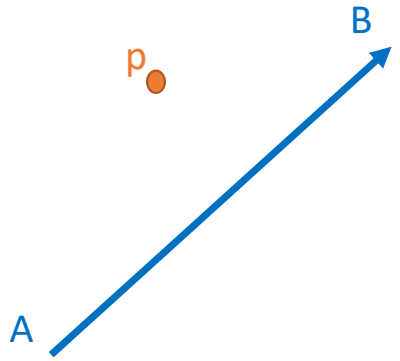
Area=0: three points on a line

Signed area of a triangle

計算幾何学の基礎(4)

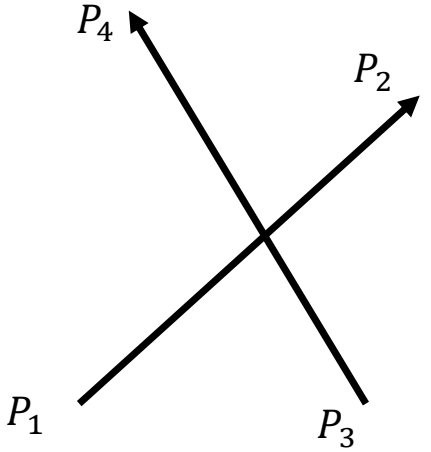
有向線分と点との位置関係

- 点 p が有向線分 AB の左側にある; Area(A,B,p)>0
- 点 p が有向線分 AB の右側にある; Area(A,B,p)<0
- 点 p が有向線分 AB の上にある; Area(A,B,p)=0



2線分の交差判定

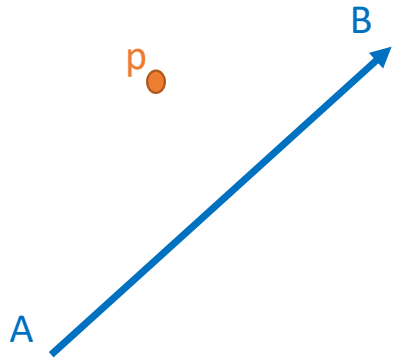
1. 線分を含む直線の式を求めて，それらの交点を求めて，それが両方の線分に含まれるかどうかを判定する
   「交点」の計算に除算を含むため，計算誤差に弱い
2. 符号付面積を用いる;
   $\Delta(P_1, P_2, P_3) \times \Delta(P_1, P_2, P_4) < 0$ かつ $\Delta(P_3, P_4, P_1) \times \Delta(P_3, P_4, P_1) < 0$
   計算は単純で誤差にも強い



Introduction to Computational Geometry (4)

Relationship between a directed line and a point

- point p is on the left side of directed line AB; Area(A,B,p)>0
- point p is on the right side of directed line AB; Area(A,B,p)<0
- point p is on directed line AB; Area(A,B,p)=0



Decision if two lines intersect or not

1. Compute equations of two lines, compute their intersection, and check if the point is on both lines.
   The computation of "intersection" requires division, so it is not robust.
2. By using signed area;
   $\Delta(P_1, P_2, P_3) \times \Delta(P_1, P_2, P_4) < 0$ and $\Delta(P_3, P_4, P_1) \times \Delta(P_3, P_4, P_1) < 0$
   The computation is simple and robust