実践的幾何アルゴリズム
Advanced Algorithms for Computational Geometry

8. 動的計画法(1)

担当:上原隆平

Advanced Algorithms for Computational Geometry
実践的幾何アルゴリズム

8. Dynamic Programming (1)

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動的計画法に基づくアルゴリズムの開発

最適化問題が対象：
制約を満たす解の中で定められた基準で最適なものを求めるのが問題.

動的計画法による問題解決

1. 最適解の構造を特徴づける.
2. 最適解の値を再帰的に定義する.
   （部分問題の解を用いて問題の解を構成する）
3. ボトムアップの形式で最適解の値を求める.
   （表を埋めていく方式）
4. 計算で求められた情報から1つの最適解を構成する.
   （最適解の値だけでなく，表を辿ることで最適解を構成）

Developing Algorithms based on Dynamic Programming

Objects: optimization problems

problem of finding an optimal solution among those satisfying given constraints.

Problem solving by dynamic programming

1. Characterize a structure of an optimal solution.
2. Define an optimal solution recursively.
   (construct a solution using solutions to subproblems)
3. Compute a value of an optimal solution in a bottom-up manner
   (in the way to fill in a table)
4. Construct an optimal solution using information obtained.
   (not only finding a value of an optimal solution but also constructing an optimal solution by following in the table)

組合せの数の計算

問題P20: n個の異なるものからk個取り出す組合せの数C(n,k)を計算せよ.

公式によると,

$C(n, k) = C(n-1, k-1) + C(n-1, k)$, $0<k<n$のとき,

$C(n, 0) = C(n, n) = 1$,

であるから，直ちに次のプログラムを得る.

```
int C(int n, int k){
  if(k==0 || k==n) return 1;
  return C(n-1, k-1) + C(n-1, k);
}
```

左のプログラムを実際に実行してみると，かなり時間がかかることがわかる.
時間がかかる原因は？

$$\left(\frac{n}{k}\right)^k \le \binom{n}{k} \le \left(\frac{ne}{k}\right)^k$$

計算時間の解析:

C(n,k)を計算するのに要する時間をT(n,k)とすると,
T(n,k)=T(n-1,k-1)+T(n-1,k)が成り立つ. よって，T(n,k)=C(n,k).
これは指数関数時間.

Number of combinations

Problem P20: Calculate the number C(n, k) of combinations to choose k items from n different items.

Using the formula,

$C(n, k) = C(n-1, k-1) + C(n-1, k)$, if $0<k<n$,

$C(n, 0) = C(n, n) = 1$.

Therefore, we have the following program.

```
int C(int n, int k){
  if(k==0 || k==n) return 1;
  return C(n-1, k-1) + C(n-1, k);
}
```

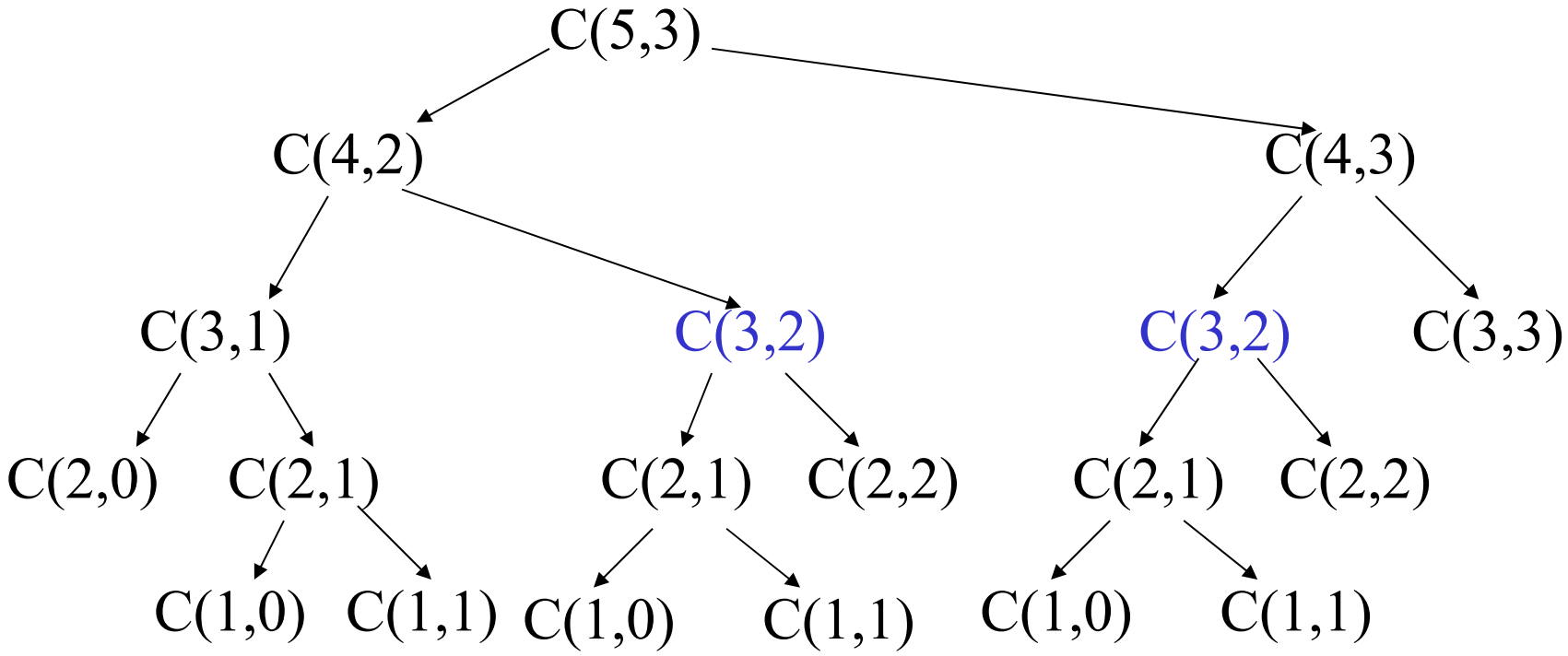
When you implement the program in practice, you will find that it takes much time. Why does it take time?

Analysis of computation time:

Let T(n,k) be time to compute C(n,k). Then, we have T(n,k)=T(n-1,k-1)+T(n-1,k). Thus, T(n,k)=C(n,k).

This is an exponential function.

プログラムの動作を調べてみよう！



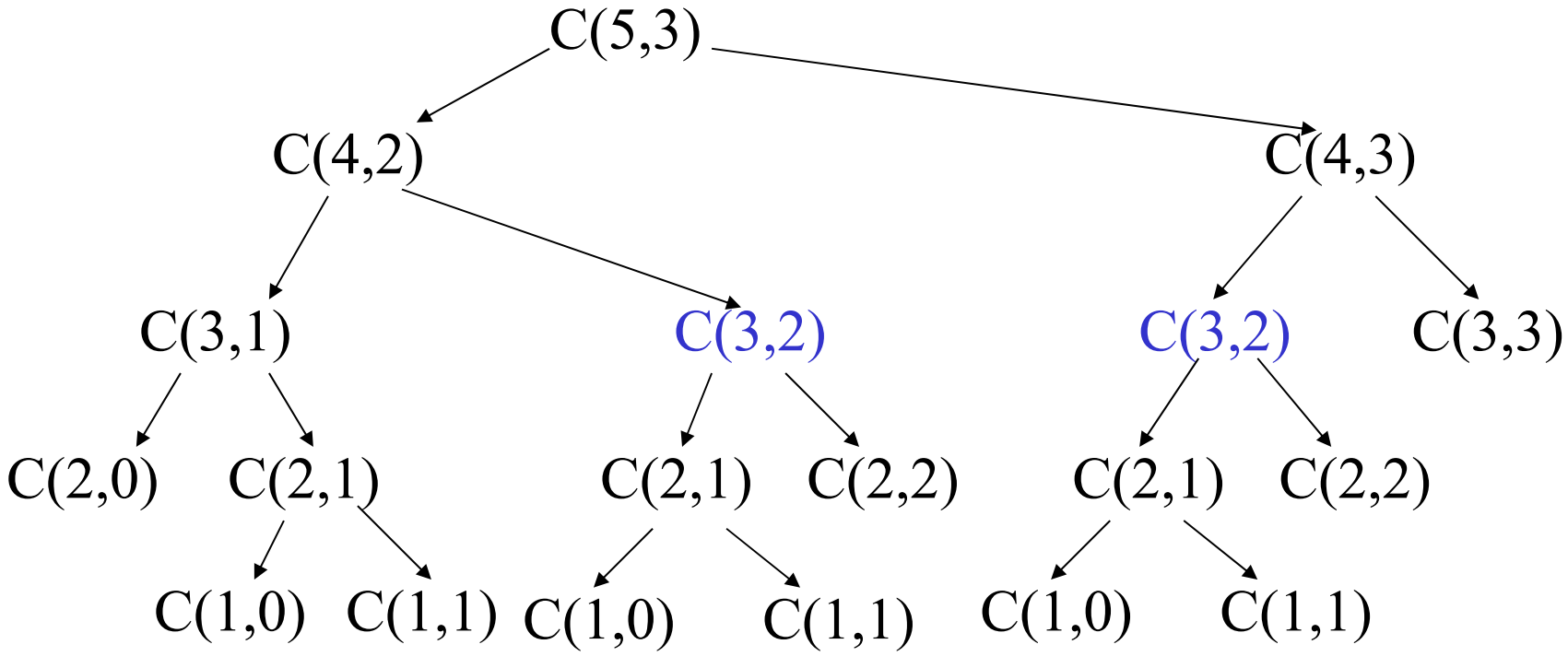
関数の呼び出しの様子

同じ値に対して関数が何度も呼び出されている.

例：C(3,2)は2回呼び出されている➔無駄

初めて(n,k)の組についてC(n,k)の値を計算するとき，その値を配列の(n,k)要素として蓄えておくと，同じ要素は2度と再計算しない.　要するに表を埋めればよい！

Let's analyze behavior of the program!



How is the function called

The function is called many times for the same value.

Example : C(3,2) is called twice. ➔ redundant

If we store the value C(n,k) as the (n,k) element of an array when it is first computed, then the same value is never computed twice. Basically, it suffices to fill in the table.

C(n,k)の表を埋めよう！

公式：C(n,k) = C(n-1,k-1) + C(n-1,k)

n-1行目の値が分かっていれば，簡単にC(n,k)が計算可能

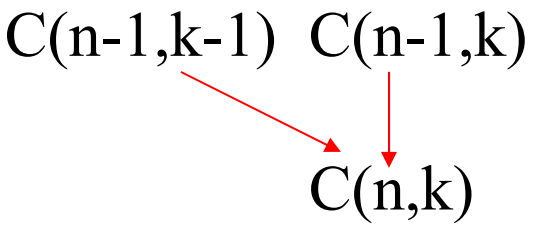
➔よって，1行目から順に埋めていけばよい．

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3						
4						
5						

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4						
5						

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5						

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1



表の各要素は定数時間で計算可能．

よって，全体の計算時間は

$O(n^2)$時間

Fill in the table C(n,k)!

Formula: C(n,k) = C(n-1,k-1) + C(n-1,k)

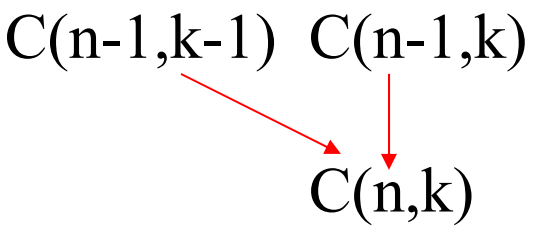
If the values in the (n-1)-st row are available, C(n,k) is easily computed. ➔Thus, we should fill in the table from the 1st row.

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3						
4						
5						

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4						
5						

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5						

	0	1	2	3	4	5
0	1					
1	1	1				
2	1	2	1			
3	1	3	3	1		
4	1	4	6	4	1	
5	1	5	10	10	5	1



Each element of the table can be computed in constant time.

Thus, the total time is

$O(n^2)$.

C言語のプログラムは下の通り:

```
int C(int n, int k){
   C[0][0]=1;
   for(i=1; i<=n; i++){
      C[i][0]=1; C[[i][i]=1;
      for(j=1; j<i; j++)
         C[i][j] = C[i-1][j-1] + C[i-1][j];
   }
   return C[n][k];
}
```

演習問題 E8-1: 素朴な方法2で、C(n,k)がオーバーフローしないならば途中でもオーバーフローしないような計算方法を考えてみよ。

素朴な方法2:

$C(n,k) = \dfrac{n!}{(n-k)!\ k!}$ であることを用いると，O(n)時間で計算可能.

ただし，この方法では整数のオーバーフローに注意.

C program is as follows:

```
int C(int n, int k){
   C[0][0]=1;
   for(i=1; i<=n; i++){
      C[i][0]=1; C[[i][i]=1;
      for(j=1; j<i; j++)
         C[i][j] = C[i-1][j-1] + C[i-1][j];
   }
   return C[n][k];
}
```

Exercise E8-1 : Consider an algorithm that computes C(n,k) based on the Naïve idea such that it computes C(n,k) correctly if C(n,k) itself does not overflow.

Naive Algorithm 2 :

Using the formula $C(n,k) = \frac{n!}{(n-k)!\ k!}$, it can be computed in O(n).

Here, note that this algorithm may suffer from numerical overflow.

フィボナッチ数の計算

問題P21: 次式で定義されるフィボナッチ数F(n)を求めよ.

$F(n) = F(n-1) + F(n-2)$, $n>1$のとき,

$F(0) = F(1) = 1$.

フィボナッチ数:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

演習問題 E8-2: 問題P20と同様の議論を展開せよ.

補足情報:

フィボナッチ数F(n)は, 黄金比$\phi=(1+\sqrt{5})/2 \fallingdotseq 1.61803$を用いて,

$F(n) = O(\phi^n)$

と表すことができる.

Computation of Fibonacci number

Problem P21: Compute the Fibonacci number F(n) defined by

$F(n) = F(n-1) + F(n-2)$, if $n>1$,

$F(0) = F(1) = 1$.

Fibonacci number:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ...

Exercise E8-2: Have an argument similar to that in Problem P20.

Supplemental information:

Using the golden ration $\phi = (1+\sqrt{5})/2 \fallingdotseq 1.61803$,

the Fibonacci number F(n) can be represented as

$F(n) = O(\phi^n)$.

最長共通部分文字列

問題P22: 長さnとmの2つの文字列AとBが与えられたとき，両方の文字列に共通する部分文字列で最長のものを求めよ.

例: A = G A A T T C A G T T A , B= G G A T C G Aのとき，GATCAは共通部分文字列である.

A= GAATTC AGTTA

B=GGA T CGA

文字列Aの任意の部分文字列A'が文字列Bの部分文字列かどうかは，A'の文字が文字列Bにおいて同じ順に出現するかどうかを調べればよい. ➔線形時間で判定可能

演習問題 E8-3:2つの文字列を入力して，最初の文字列が2番目の文字列の部分文字列になっているかどうかを線形時間で判定するプログラムを書け.

Longest Common Subsequence

Problem P22: Given two strings A and B of lengths n and m, find the longest substring common to both of them.

Example: For A = G A A T T C A G T T A and B= G G A T C G A, the longest common substring is GATCA.

A= GAATTC AGTTA

B=GGA T CGA

Any substring A' of A is a substring of B if characters of A' appear in the same order in the string B.

➔It can be determined in linear time.

Exercise E8-3: Write a program to determine whether the first string of two input strings is a substring of the second string in linear time.

アルゴリズムP22-A0:（腕力法）
文字列Aのすべての部分文字列A'について，A'が文字列Bの部分文字列になっているかどうかを確かめ，最も長い共通文字列を最後に出力する.

計算時間の解析:
- 長さnの文字列の部分文字列は全部で2^n通りある.
- この文字列が文字列Bより長いときは，明らかに文字列Bの部分文字列ではない.
- そうでないとき，それぞれにO(m)の時間がかかる.
- よって，全体の計算時間は，$O(2^n m)$時間となる.

高速化は可能か？
多項式時間のアルゴリズムは存在するか？

Algorithm P22-A0:（Brute-Force Algorithm）
For each substring A' of a string A, determine whether A' is a substring of a string B, and finally output the longest common substring.

Analysis of computation time：

- There are 2^n different substrings of a string of length n.
- If this substring is longer than the string B, obviously it is not a substring of B.
- Otherwise, each test takes O(m) time.
- Thus, the total time is $O(2^n m)$ time.

Is it possible to have faster algorithm?
Is there any polynomial-time algorithm?

アルゴリズムP22-A1:

$A= a_1a_2...a_n$, $B= b_1b_2... b_m$
$L[i,j] = a_1a_2...a_i$と$b_1b_2... b_j$の最長共通部分文字列の長さ

観察:
(0) i=0またはj=0のとき，$L[i,j]=0$.
(1) $a_i = b_j$ → $L[i,j] = L[i-1,j-1]+1$
(2) $a_i \neq b_j$ → $L[i,j] = \max\{ L[i,j-1], L[i-1, j]\}$

A=GAATTC AGTTA
B= GGATC GA
$a_i = b_j$のとき

A=GAATTC AGTTA
B=GGATCG A
$a_i \neq b_j$のとき

したがって，今度も表L[i,j]を順に埋めていけばよい！
表のサイズはn×mだから，計算時間もO(nm)時間.

Algorithm P22-A1:

$A= a_1a_2...a_n$, $B= b_1b_2... b_m$

L[i,j] = the length of the longest substring common to $a_1a_2...a_i$ and $b_1b_2... b_j$

Observation:

(0) if i=0 or j=0, L[i,j]=0.

(1) $a_i = b_j \rightarrow$ L[i,j] = L[i-1,j-1]+1

(2) $a_i \neq b_j \rightarrow$ L[i,j] = max{ L[i,j-1], L[i-1, j]}

A=GAATTC AGTTA
B= GGATC GA
when $a_i = b_j$

A=GAATTC AGTTA
B=GGATCG A
when $a_i \neq b_j$

Therefore, it suffices to fill in the table L[i,j] in order.
Since the table size is n × m, it takes O(nm) time.

アルゴリズムP22-A1:

```
for(i=0; i<=n; i++)
   L[i][0]=0;
for(j=0; j<=m; j++)
   L[0][ j] = 0;
for(i=1; i<=n; i++)
   for(j=1; j<=m; j++)
      if( a[i] == b[j]) L[i][j] = L[i-1][j-1]+1;
      else L[i][j] = max{ L[i][j-1], L[i-1][j] };
return L[n][m];
```

A=XYXXZXYZXY,
B=ZXZYYZXXYXXZ
のときの動作例

A= X Y XXZXYZXY,
B=ZXZYYZXXYXXZ

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	1	1	1	1	1
2	0	0	1	1	2	2	2	2	2	2	2	2	2
3	0	0	1	1	2	2	2	3	3	3	3	3	3
4	0	0	1	1	2	2	2	3	4	4	4	4	4
5	0	1	1	2	2	2	3	3	4	4	4	4	5
6	0	1	2	2	2	2	3	4	4	4	5	5	5
7	0	1	2	2	3	3	3	4	4	5	5	5	5
8	0	1	2	3	3	3	4	4	4	5	5	5	6
9	0	1	2	3	3	3	4	4	5	5	6	6	6
10	0	1	2	3	4	4	4	5	5	6	6	6	6

Algorithm P22-A1:

```
for(i=0; i<=n; i++)
   L[i][0]=0;
for(j=0; j<=m; j++)
   L[0][ j] = 0;
for(i=1; i<=n; i++)
   for(j=1; j<=m; j++)
      if( a[i] == b[j]) L[i][j] = L[i-1][j-1]+1;
      else L[i][j] = max{ L[i][j-1], L[i-1][j] };
return L[n][m];
```

Example for the case
A=XYXXZXYZXY and
B=ZXZYYZXXYXXZ

A= X Y XXZXYZXY,
B=ZXZYYZXXYXXZ

	0	1	2	3	4	5	6	7	8	9	10	11	12
0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	1	1	1	1	1	1	1	1	1	1	1
2	0	0	1	1	2	2	2	2	2	2	2	2	2
3	0	0	1	1	2	2	2	3	3	3	3	3	3
4	0	0	1	1	2	2	2	3	4	4	4	4	4
5	0	1	1	2	2	2	3	3	4	4	4	4	5
6	0	1	2	2	2	2	3	4	4	4	5	5	5
7	0	1	2	2	3	3	3	4	4	5	5	5	5
8	0	1	2	3	3	3	4	4	4	5	5	5	6
9	0	1	2	3	3	3	4	4	5	5	6	6	6
10	0	1	2	3	4	4	4	5	5	6	6	6	6

最適解の構成

最適解の値は表を埋めることにより求められるが，その値を達成する最適解はどのように構成すればよいか？

最長共通部分文字列の問題では最長の共通部分文字列の長さ（最適解の値）だけでなく，文字列そのもの（最適解）も求めたい．

表を埋めるとき，L[i][j]の値が同じ表のどの値によって決まったかを記録する：

(1) $a_i = b_j$ → L[i,j] = L[i-1,j-1]+1
(i-1, j-1)を記憶
(2) $a_i \neq b_j$ → L[i,j] = max{ L[i,j-1], L[i-1, j]}
L[i,j-1]>L[i-1,j]のとき，(i, j-1)を記憶，
そうでないとき，(i-1, j)を記憶

Construction of an optimal solution

The value of an optimal solution is obtained by filling in the table. How can we construct an optimal solution achieving the value?

In the problem of finding the longest common substring, we want to find not only the length (value of an optimal solution) but also the longest such substring (optimal solution) itself.

When we fill in the table, we memorize which table element determined L[i][j].

(1) $a_i = b_j \rightarrow L[i,j] = L[i-1,j-1]+1$
(i-1, j-1) is memorized
(2) $a_i \neq b_j \rightarrow L[i,j] = \max\{ L[i,j-1], L[i-1, j]\}$
if L[i,j-1]>L[i-1,j] then (i, j-1) is memorized, and
otherwise, (i-1, j) is memorized.

具体的なプログラム

```
for(i=1; i<n; i++)
  for(j=1; j<m; j++){
    if( A[i] == B[j] ){
      L[i][j] = L[i-1][j-1] + 1;
      B1[i][j] = i-1; B2[i][j] = j-1;
    } else {
      L[i][j] = max2(L[i][j-1], L[i-1][j]);
      if( L[i][j-1] > L[i-1][j] ){
        L[i][j] = L[i][j-1];
        B1[i][j] = B1[i][j-1]; B2[i][j] = B2[i][j-1];
      } else {
        L[i][j] = L[i-1][j];
        B1[i][j] = B1[i-1][j]; B2[i][j] = B2[i-1][j];
      }
    }
}
```

演習問題 E8-4：実際にプログラムを作って動作を確認せよ.

Concrete program

```
for(i=1; i<n; i++)
  for(j=1; j<m; j++){
    if( A[i] == B[j] ){
      L[i][j] = L[i-1][j-1] + 1;
      B1[i][j] = i-1; B2[i][j] = j-1;
    } else {
      L[i][j] = max2(L[i][j-1], L[i-1][j]);
      if( L[i][j-1] > L[i-1][j] ){
        L[i][j] = L[i][j-1];
        B1[i][j] = B1[i][j-1]; B2[i][j] = B2[i][j-1];
      } else {
        L[i][j] = L[i-1][j];
        B1[i][j] = B1[i-1][j]; B2[i][j] = B2[i-1][j];
      }
    }
}
```

Exercise E8-4 : Write a program in practice to see its behavior.

A=XYXXZXYZXY, B=ZXZYYZXXYXXZの場合

バックトラック用の表

	1	2	3	4	5	6	7	8	9	10	11	12
1	(0,0)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,6)	(0,7)	(0,7)	(0,9)	(0,10)	(0,10)
2	(0,0)	(0,1)	(0,1)	(1,3)	(1,4)	(1,4)	(1,4)	(1,4)	(1,8)	(1,8)	(1,8)	(1,8)
3	(0,0)	(2,1)	(0,1)	(1,3)	(1,4)	(1,4)	(2,6)	(2,7)	(2,7)	(2,9)	(2,10)	(2,10)
4	(0,0)	(3,1)	(0,1)	(1,3)	(1,4)	(1,4)	(3,6)	(3,7)	(3,7)	(3,9)	(3,10)	(3,10)
5	(4,0)	(3,1)	(4,2)	(1,3)	(1,4)	(4,5)	(3,6)	(3,7)	(3,7)	(3,9)	(3,10)	(4,11)
6	(4,0)	(5,1)	(4,2)	(1,3)	(1,4)	(4,5)	(5,6)	(5,7)	(3,7)	(5,9)	(5,10)	(4,11)
7	(4,0)	(5,1)	(4,2)	(6,3)	(6,4)	(4,5)	(5,6)	(5,7)	(6,8)	(5,9)	(5,10)	(4,11)
8	(7,0)	(5,1)	(7,2)	(6,3)	(6,4)	(7,5)	(5,6)	(5,7)	(6,8)	(5,9)	(5,10)	(7,11)
9	(7,0)	(8,1)	(7,2)	(6,3)	(6,4)	(7,5)	(8,6)	(8,7)	(6,8)	(8,9)	(8,10)	(7,11)
10	(7,0)	(8,1)	(7,2)	(9,3)	(9,4)	(7,5)	(8,6)	(8,7)	(9,8)	(8,9)	(8,10)	(7,11)

L[10][12]から逆に辿ると

L[10][12] →L[7][11]→L[5][10]→L[3][9]→L[2][7]→L[1][4]→L[0][1]

a[8]b[12] a[6]b[11] a[4]b[10] a[3]b[8] a[2]b[5] a[1]b[2]

ゆえに，最長共通部分文字列は

123456789A 123456789ABC

XYXXZXYZXY ZXZYYZXXYXXZ XYXXXZ

For the case: A=XYXXZXYZXY, B=ZXZYYZXXYXXZ

Table for backtrack

	1	2	3	4	5	6	7	8	9	10	11	12
1	(0,0)	(0,1)	(0,1)	(0,1)	(0,1)	(0,1)	(0,6)	(0,7)	(0,7)	(0,9)	(0,10)	(0,10)
2	(0,0)	(0,1)	(0,1)	(1,3)	(1,4)	(1,4)	(1,4)	(1,4)	(1,8)	(1,8)	(1,8)	(1,8)
3	(0,0)	(2,1)	(0,1)	(1,3)	(1,4)	(1,4)	(2,6)	(2,7)	(2,7)	(2,9)	(2,10)	(2,10)
4	(0,0)	(3,1)	(0,1)	(1,3)	(1,4)	(1,4)	(3,6)	(3,7)	(3,7)	(3,9)	(3,10)	(3,10)
5	(4,0)	(3,1)	(4,2)	(1,3)	(1,4)	(4,5)	(3,6)	(3,7)	(3,7)	(3,9)	(3,10)	(4,11)
6	(4,0)	(5,1)	(4,2)	(1,3)	(1,4)	(4,5)	(5,6)	(5,7)	(3,7)	(5,9)	(5,10)	(4,11)
7	(4,0)	(5,1)	(4,2)	(6,3)	(6,4)	(4,5)	(5,6)	(5,7)	(6,8)	(5,9)	(5,10)	(4,11)
8	(7,0)	(5,1)	(7,2)	(6,3)	(6,4)	(7,5)	(5,6)	(5,7)	(6,8)	(5,9)	(5,10)	(7,11)
9	(7,0)	(8,1)	(7,2)	(6,3)	(6,4)	(7,5)	(8,6)	(8,7)	(6,8)	(8,9)	(8,10)	(7,11)
10	(7,0)	(8,1)	(7,2)	(9,3)	(9,4)	(7,5)	(8,6)	(8,7)	(9,8)	(8,9)	(8,10)	(7,11)

If we trace the table from L[10][12] in reverse order,

L[10][12] →L[7][11]→L[5][10]→L[3][9]→L[2][7]→L[1][4]→L[0][1]

a[8]b[12] a[6]b[11] a[4]b[10] a[3]b[8] a[2]b[5] a[1]b[2]

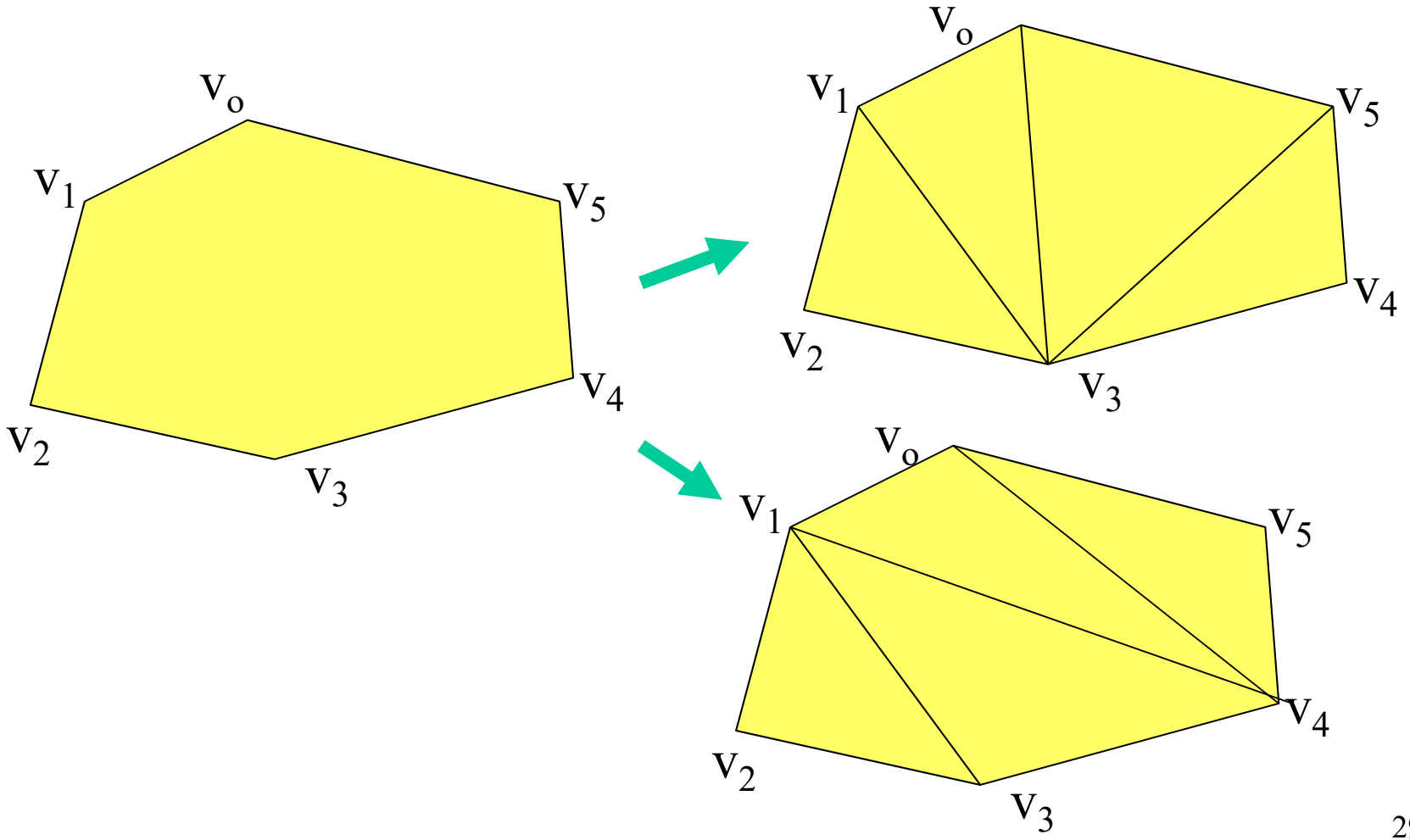
Thus, the longest common substring is

123456789A 123456789ABC

XYXXZXYZXY ZXZYYZXXYXXZ XYXXXZ

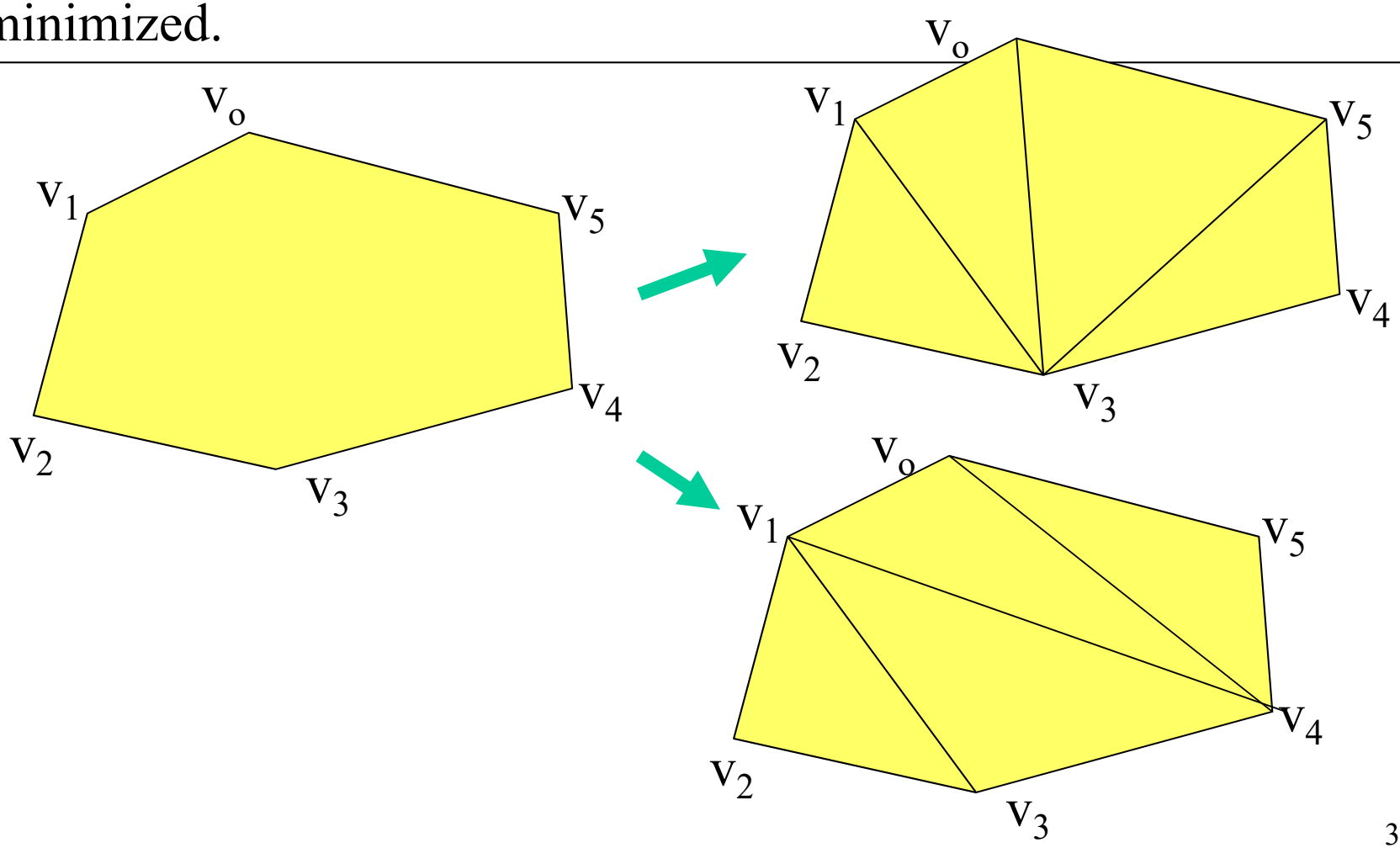
問題P23:（最適三角形分割）

凸多角形が頂点の系列として与えられたとき，2つの頂点の間に弦を引くことにより多角形の内部を三角形に分割することができるが，弦の長さの総和を最小にする三角形分割を求めよ.



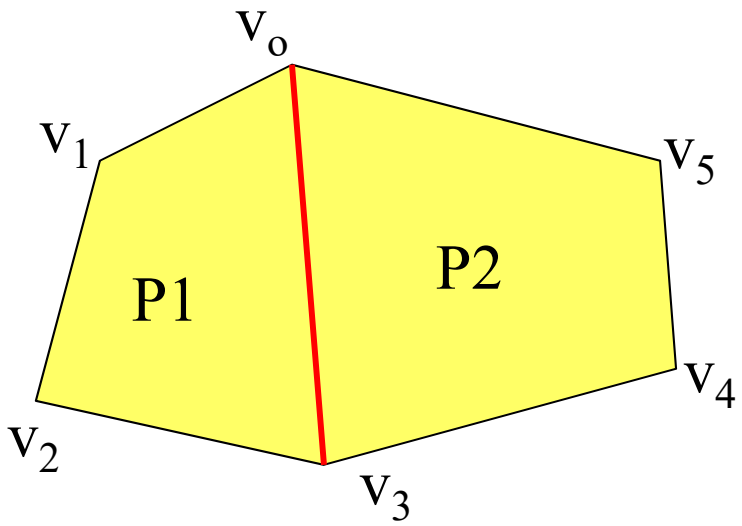
Problem P23:（Optimal triangulation）

Given a convex polygon as a vertex sequence, we can partition its interior into triangles by drawing chords between two vertices. Find a triangulation so that the total sum of lengths of chords is minimized.



最適解の構造を特徴づけ，最適解の値を再帰的に定義する．

凸多角形を$P(v_o, v_1, v_2, v_3, v_4, v_5)$のように頂点の系列で表現．
頂点v_oについて考えると，
　ケース１：v_oから別の頂点v_iにむけて弦を引く．
　ケース２：v_oにつながる弦はない．



演習問題E9-1：ケース２のとき，必ず両隣の頂点が弦で結ばれることを証明せよ．

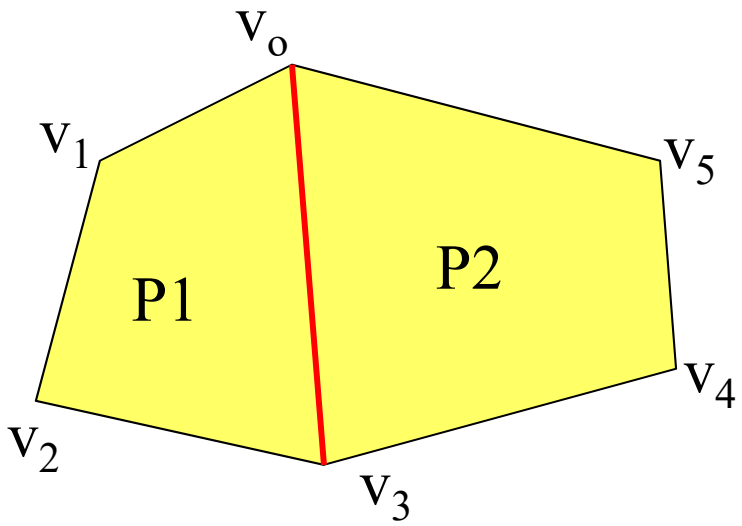
弦を１本引くことによって生じる部分多角形はやはり凸であり，頂点数はn未満だから，すべての部分多角形について最適解を求めておけば，元の問題の最適解が得られる．

Characterize structure of an optimal solution and define the value of an optimal solution recursively.

Represent a convex polygon as a vertex sequence like $P(v_o, v_1, v_2, v_3, v_4, v_5)$. Considering a vertex v_o,

Case 1:draw a chord from v_o to another vertex v_i.

Case 2: there is no chord incident to v_o.



Exercise E9-1 : Prove that two adjacent must be connected by a chord in Case 2.

When we draw a chord, two resulting subpolygons are convex. Since it has less than n vertices, if we have all optimal solutions for all subproblems, then we can obtain an optimal solution.

凸n角形を分割する仕方は何通りあるだろう？

凸n角形$(v_0, \dots, v_{n-1})$を分割する仕方がf(n)通りあるとする.

ケース1：v_oから別の頂点v_iにむけて弦を引く.

弦としては$(v_0,v_2), (v_0,v_3), \dots, (v_0,v_{n-2})$が考えられる.

弦(v_0,v_2)→3角形(v_0,v_1,v_2) + (n-1)角形$(v_0,v_2,v_3,\dots,v_{n-1})$

弦(v_0,v_3)→4角形$(v_0,v_1,v_2,,v_3)$ + (n-2)角形$(v_0,v_3,v_4,\dots,v_{n-1})$

弦(v_0,v_4)→5角形$(v_0,v_1,v_2,,v_3,v_4)$ + (n-3)角形$(v_0,v_4,v_5,\dots,v_{n-1})$

：

弦(v_0,v_{n-2})→(n-1)角形$(v_0,v_1,v_2,\dots,v_{n-2})$ + 3角形(v_0,v_{n-2},v_{n-1})

ケース2：v_oにつながる弦はない.

弦(v_1,v_{n-1})→3角形(v_0,v_1,v_{n-1}) + (n-1)角形$(v_1,v_2,v_3,\dots,v_{n-1})$

同じ三角形分割が何度も現れることを考えると

$f(n) \leqq 3f(n-1) + 2f(n-2) + 2f(n-3) + \cdots + 2f(4) + 3f(3)$

$f(3) = 1$.

$g(n)=2g(n-1)$, $g(3)=1$なら$g(n)=2^{n-3}$だから，f(n)も指数関数.

すなわち，この方法では多項式時間では解けない！

How many different triangulations of a convex polygon?

Suppose that there are f(n) ways to triangulate a convex polygon $(v_0, \dots, v_{n-1})$.

Case 1: Draw a chord from v_o to v_i.

possible chords are (v_0,v_2), (v_0,v_3), ... , (v_0,v_{n-2}).

(v_0,v_2)=>triangle (v_0,v_1,v_2) + (n-1)-gon$(v_0,v_2,v_3,\dots,v_{n-1})$

(v_0,v_3)=>quadrangle$(v_0,v_1,v_2,,v_3)$ + (n-2)-gon$(v_0,v_3,v_4,\dots,v_{n-1})$

(v_0,v_4)=>pentagon$(v_0,v_1,v_2,,v_3,v_4)$ + (n-3)-gon$(v_0,v_4,v_5,\dots,v_{n-1})$

:

(v_0,v_{n-2})=>(n-1)-gon$(v_0,v_1,v_2,\dots,v_{n-2})$ + triangle(v_0,v_{n-2},v_{n-1})

Case 2: there is no chord incident to v_o.

(v_1,v_{n-1})=>triangle (v_0,v_1,v_{n-1}) + (n-1)-gon$(v_1,v_2,v_3,\dots,v_{n-1})$

Considering duplicate appearance of triangles,

$f(n) \leqq 3f(n-1) + 2f(n-2) + 2f(n-3) + \cdots + 2f(4) + 3f(3)$

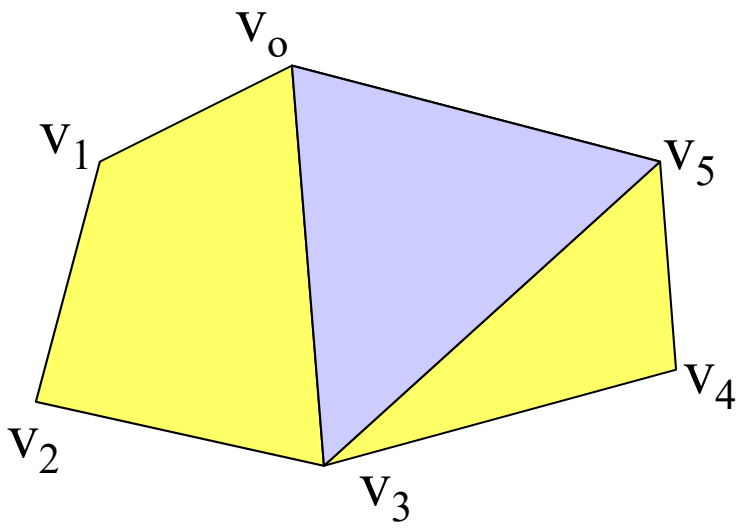
$f(3) = 1$.

$g(n)=2g(n-1)$, $g(3)=1$ then $g(n)=2^{n-3}$, thus f(n) is also exponential.

That is, this method does not lead to polynomial-time algorithm.

別の方法で再帰的に解を表現する.

凸多角形P(v_o, v_1, v_2, v_3, v_4, v_5)の辺(v_o, v_5)は必ずどれかの三角形に含まれなければならない. そのような三角形は(v_o, v_1, v_5),(v_o, v_2, v_5),(v_o, v_3, v_5),(v_o, v_4, v_5)の中の一つ.

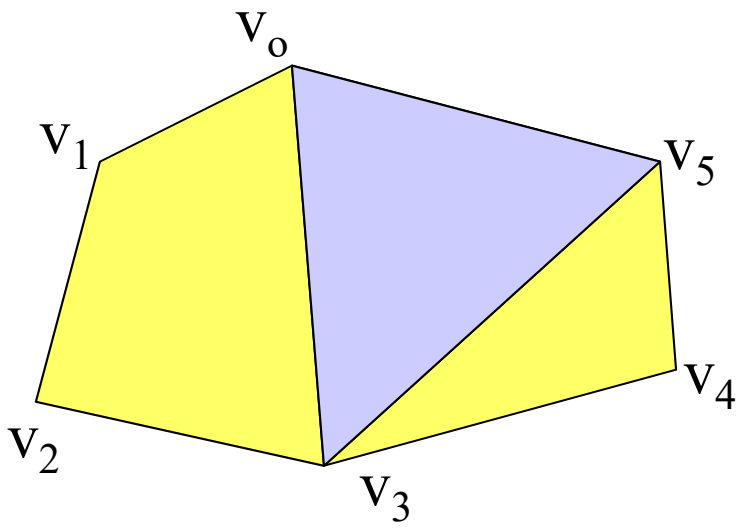


三角形(v_o, v_3, v_5)の場合,
残りの部分は2つの凸多角形に分割される.

一般に, 凸多角形P(v_o, ..., v_{n-1})を辺(v_o, v_{n-1})を含む三角形(v_o, v_k, v_{n-1})によって分割すると,
凸多角形P(v_o,v_1, ..., v_k)と凸多角形P(v_k,v_{k+1}, ..., v_{n-1})に分かれる.
これは, [0,n-1]という区間を[0,k]と[k,n-1]に分割することに対応.
よって, 分割の仕方は部分区間の数, $O(n^2)$通りしかない!

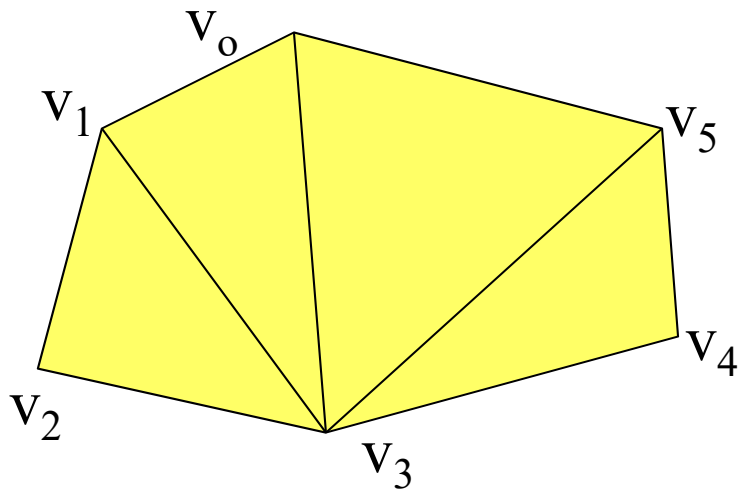
Recursive representation of a solution in a different way

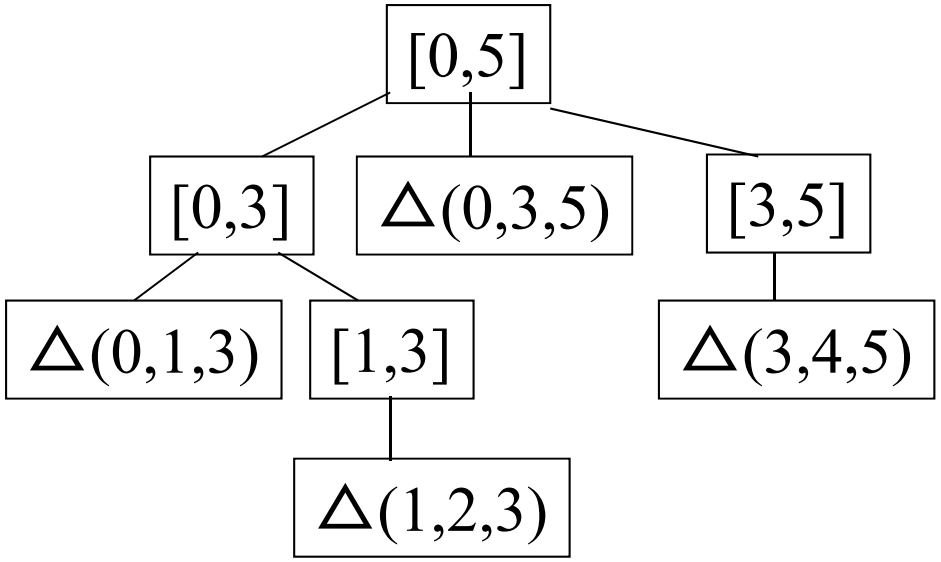
The edge (v_o, v_5) of the convex polygon $P(v_o, v_1, v_2, v_3, v_4, v_5)$ must be included in at least one triangle. Such a triangle is one of (v_o, v_1, v_5),(v_o, v_2, v_5),(v_o, v_3, v_5), and (v_o, v_4, v_5).

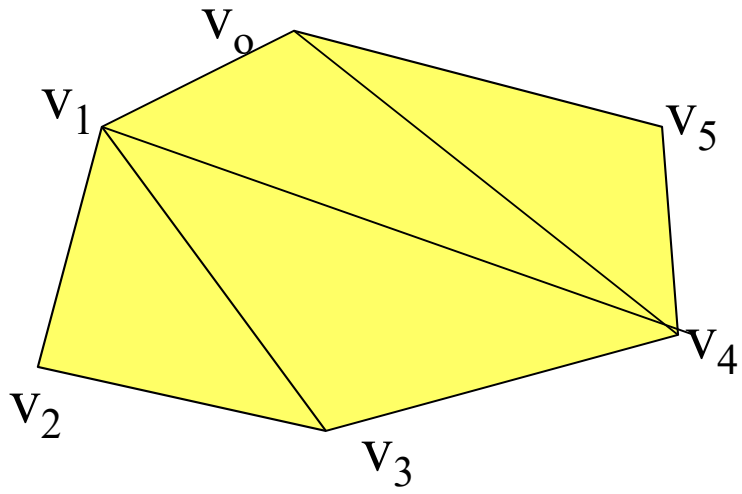


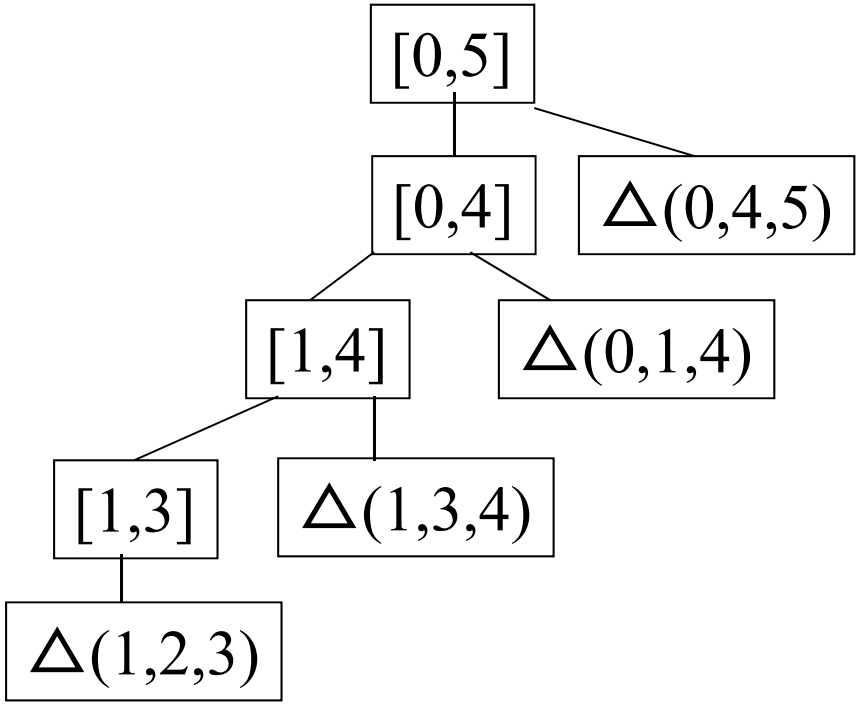
For the triangle (v_o, v_3, v_5), the remaining part is partitioned into two convex polygons.

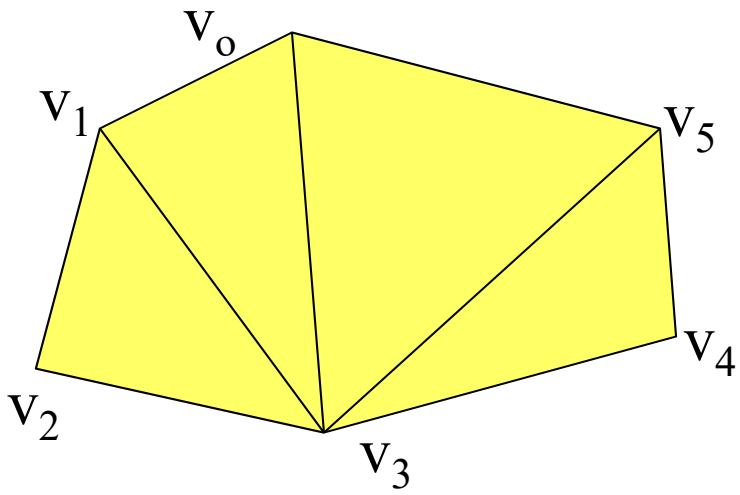
In general, when we partition the polygon $P(v_o, ..., v_{n-1})$ by a triangle containing the edge (v_o, v_{n-1}), we have two convex polygons: $P(v_o,v_1, ..., v_k)$ and $P(v_k,v_{k+1}, ..., v_{n-1})$. This corresponds to a partition of an interval [0,n-1] into [0,k] and [k,n-1]. Thus, the number of different partitions is that of different intervals, that is, $O(n^2)$.

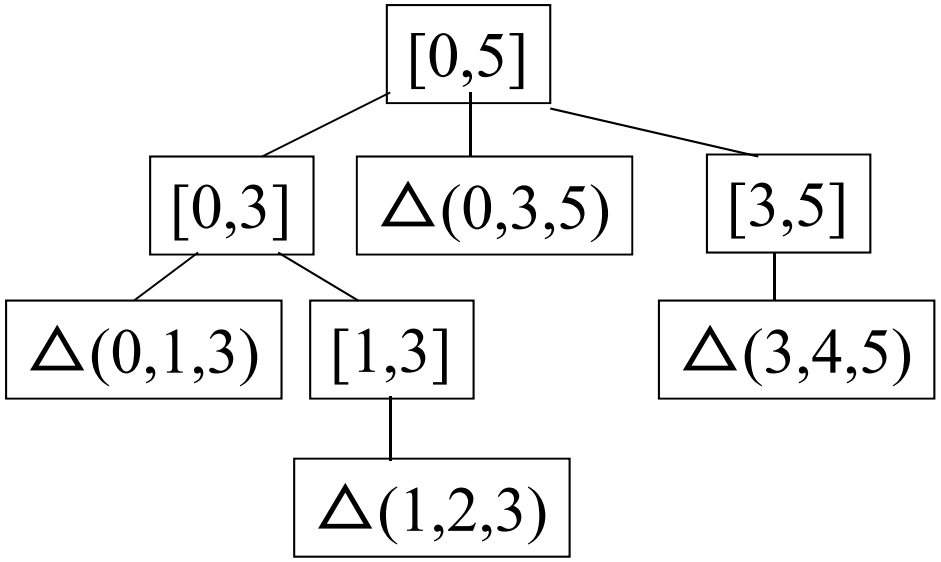

v_o
v_1
v_5
v_4
v_2
v_3

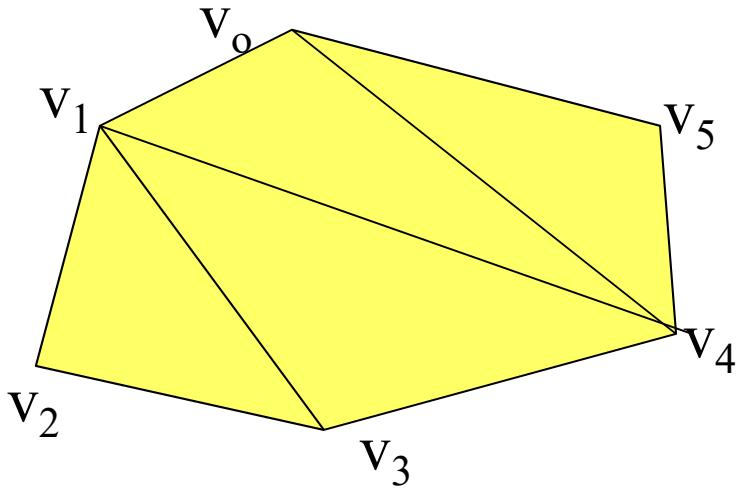

[0,5]
[0,3]
△(0,3,5)
[3,5]
△(0,1,3)
[1,3]
△(3,4,5)
△(1,2,3)

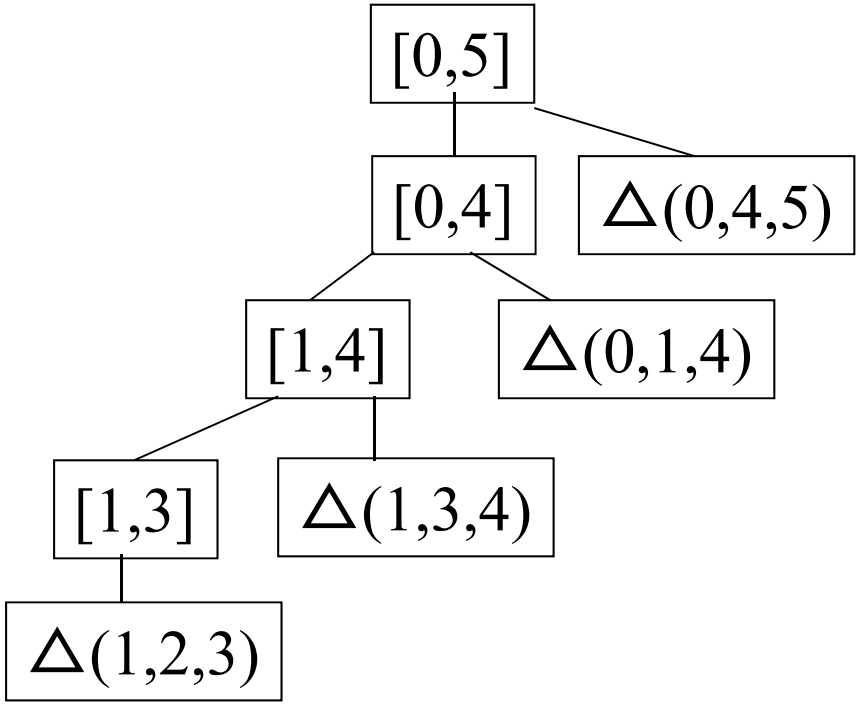

v_o
v_1
v_5
v_4
v_2
v_3


[0,5]
[0,4]
△(0,4,5)
[1,4]
△(0,1,4)
[1,3]
△(1,3,4)
△(1,2,3)

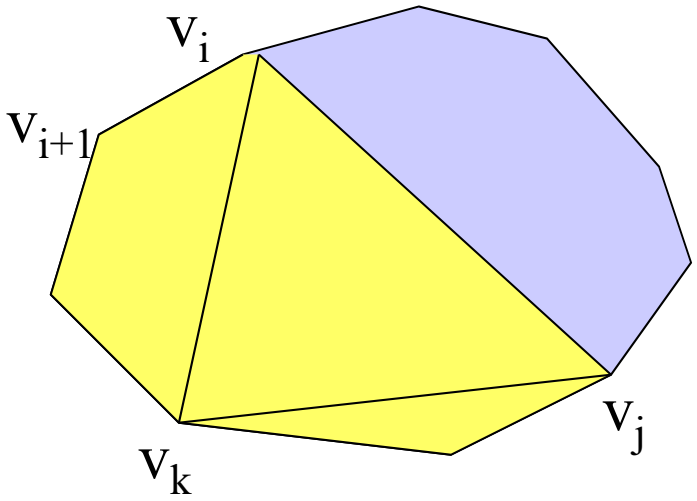

v_o
v_1
v_5
v_4
v_2
v_3


[0,5]
[0,3]
△(0,3,5)
[3,5]
△(0,1,3)
[1,3]
△(3,4,5)
△(1,2,3)


v_o
v_1
v_5
v_4
v_2
v_3


[0,5]
[0,4]
△(0,4,5)
[1,4]
△(0,1,4)
[1,3]
△(1,3,4)
△(1,2,3)

[0,n-1]の部分区間[i,j]に対応する多角形の分割



凸多角形$P(v_i, v_{i+1}, \ldots, v_j)$を
　三角形(v_i, v_k, v_j),
　凸多角形$(v_i, v_{i+1}, \ldots, v_k)$
　凸多角形$(v_k, v_{k+1}, \ldots, v_j)$
に分割.
ただし, $k=i+1$のときと$k=j-1$のときは
三角形と残りの多角形の2つに分割.

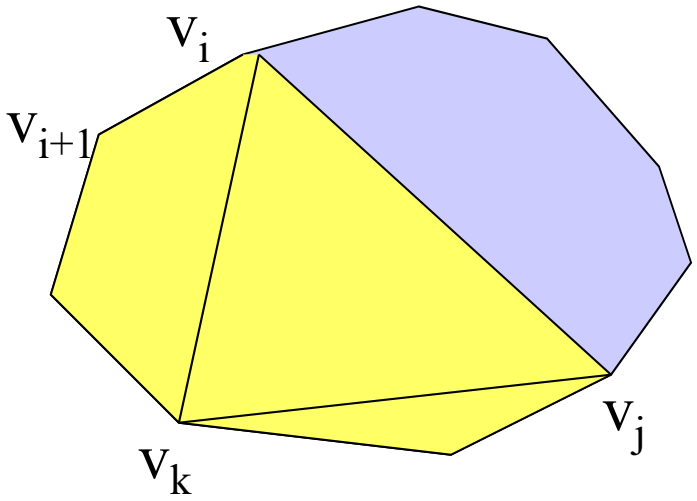
$L[i,j]$ = 頂点列$(v_i, v_{i+1}, \ldots, v_j)$によって定まる部分多角形の内部に含まれる弦の長さの総和.

$w[i,j]$ = 頂点v_iと頂点v_jを結ぶ弦の長さ

とすると, 漸化式は

$$L[i,j] = \min\{ L[i+1,j]+w[i+1,j],$$
$$\min\{ L[i,k]+L[k,j]+w[i,k]+w[k,j] \mid k=i+2, \ldots, j-2\},$$
$$L[i,j-1]+w[i,j-1]\}$$

Partition of a polygon corresponding to a subinterval [i,j] of [0,n-1]



Partition a convex polygon $P(v_i,v_{i+1}, ..., v_j)$ into
  a triangle (v_i, v_k, v_j),
  a convex polygon$(v_i,v_{i+1}, ..., v_k)$ and
  a convex polygon$(v_k,v_{k+1}, ..., v_j)$,
if $k>i+1$ and $k<j-1$, and
  a triangle and a convex polygon
if $k=i+1$ or $k=j-1$.

L[i,j] = the sum of lengths of chords contained in the interior of the subpolygon determined by vertex sequence $(v_i,v_{i+1}, ..., v_j)$.
w[i,j] = the length of the chord between vertices v_i and v_j
Then, we have the following recurrence equation:

$$L[i,j] = \min\{ L[i+1,j]+w[i+1,j], \min\{ L[i,k]+L[k,j]+w[i,k]+w[k,j] \mid k=i+2, \ldots, j-2\}, L[i,j-1]+w[i,j-1]\}$$

アルゴリズムP23-A0:

弦の長さの総和を最小にする三角形分割

入力：凸多角形$P(v_0, v_1, \ldots, v_{n-1})$

出力：最適な三角形分割における弦の長さの総和

```
for(i=0; i<n; i++)
   for(j=i+2; j<n; j++){
      w[i,j] = 頂点v_iと頂点v_jを結ぶ弦の長さ;
      L[i,j] = 0;}
for(d=3; d<n; d++)
   for(i=0; i<n; i++)
      for(j=i+d; j<n; j++){
         msf = min( L[i+1,j]+w[i+1,j], L[i,j-1]+w[i,j-1]);
         for(k=i+2; k<=j-2; k++)
            if( L[i,k]+L[k,j]+w[i,k]+w[k,j] < msf)
               msf = L[i,k]+L[k,j]+w[i,k]+w[k,j];
      }
return L[0,n-1];
```

Algorithm P23-A0:

Triangulation to minimize the sum of lengths of chords

Input : convex polygon $P(v_0, v_1, ..., v_{n-1})$

Output : the sum of lengths of chords in an optimal triangulation

```
for(i=0; i<n; i++)
   for(j=i+2; j<n; j++){
      w[i,j] = the length of the chord between vertices v_i and v_j
L[i,j] = 0;}
for(d=3; d<n; d++)
   for(i=0; i<n; i++)
      for(j=i+d; j<n; j++){
         msf = min( L[i+1,j]+w[i+1,j], L[i,j-1]+w[i,j-1]);
         for(k=i+2; k<=j-2; k++)
            if( L[i,k]+L[k,j]+w[i,k]+w[k,j] < msf)
               msf = L[i,k]+L[k,j]+w[i,k]+w[k,j];
      }
return L[0,n-1];
```

演習問題E9-2：問題P24では弦の長さの総和を最小にする三角形分割を求めたが，弦の長さの２乗和を最小にする場合はどうか．

演習問題E9-3：各三角形の面積の２乗和を最小にする場合はどうか．

演習問題E9-4：各三角形の面積の和を最小にする場合はどうか．

演習問題E9-5：最適解の値だけではなく，最適解（最適三角形分割）そのものを求めるようにアルゴリズムを変更せよ．

おまけ：

Z. Abel, E. D. Demaine, M. Demaine, H. Ito, J. Snoeyink and R. Uehara: Bumpy Pyramid Folding,
The 26th Canadian Conference on Computational Geometry (CCCG 2014), pp. 258-266, 2014/08/11-2014/08/13, Halifax, Canada.

Exercise E9-2 : In Problem P24 we tried to find a triangulation to minimize the sum of chord lengths. What about the case where the sum of squared chord lengths is minimized?

Exercise E9-3 : What about the case where the sum of squared area of triangles?

Exercise E9-4 : What about if we want to minimize the sum of area of triangles?

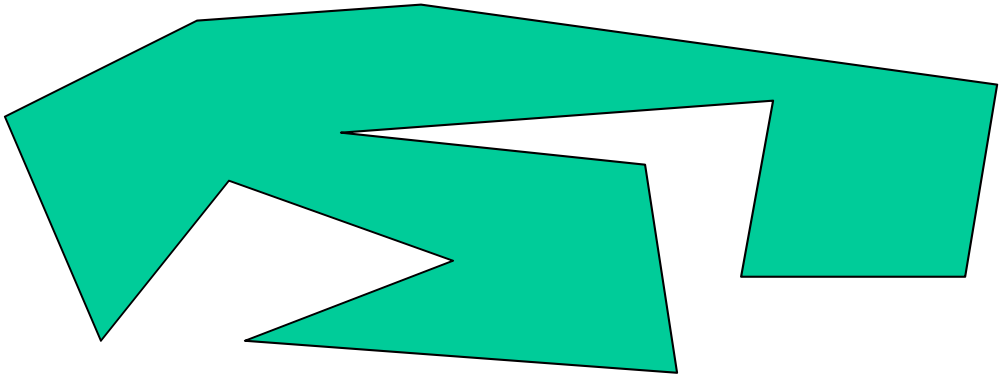
Exercise E9-5 : Modify the algorithm so that not only the value of an optimal solution but also an optimal solution itself (optimal triangulation) is obtained.

Add. :

Z. Abel, E. D. Demaine, M. Demaine, H. Ito, J. Snoeyink and R. Uehara: Bumpy Pyramid Folding,
The 26th Canadian Conference on Computational Geometry (CCCG 2014), pp. 258-266, 2014/08/11-2014/08/13, Halifax, Canada.

凸多角形ではなく，一般の多角形の三角形分割の場合に拡張できるだろうか？

違いは，凸多角形の場合にはどの２点間にも弦を引けたが，一般の多角形では弦を引けないことがある．



w[i,j]の定義を変更する：

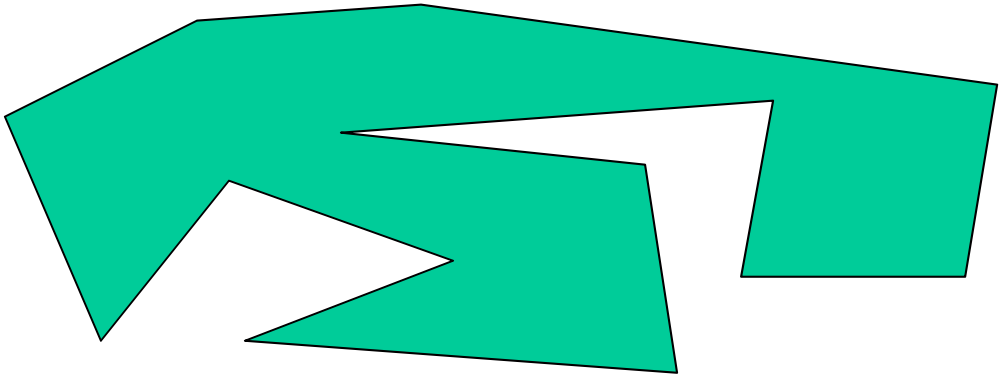
頂点v_iと頂点v_jを結ぶ線分が多角形の内部だけを通るときその長さをw[i,j]とし，そうでないときは，∞とする．

後は，まったく同じアルゴリズムで最適解を求めることができる．

演習問題E9-6：頂点v_iと頂点v_jを結ぶ線分が多角形の内部だけを通るかどうかを判定する方法を考えよ．

Is it possible to extend it to the case of triangulation of a general polygon instead of a convex polygon?

One difference is that we could connect any two vertices as chords in a convex polygon but a general polygon may not allow it.



Modify the definition of w[i,j] :

Only if the segment between vertices v_i and v_j is contained in the interior of the polygon, its length is defined to be w[i,j], Otherwise, w[i,j] is ∞.

Only with this modification we can find an optimal solution.

Exercise E9-6 : Devise a method for determining whether a segment between vertices v_i and v_j is contained in the interior of a polygon.