Introduction to Computational Origami

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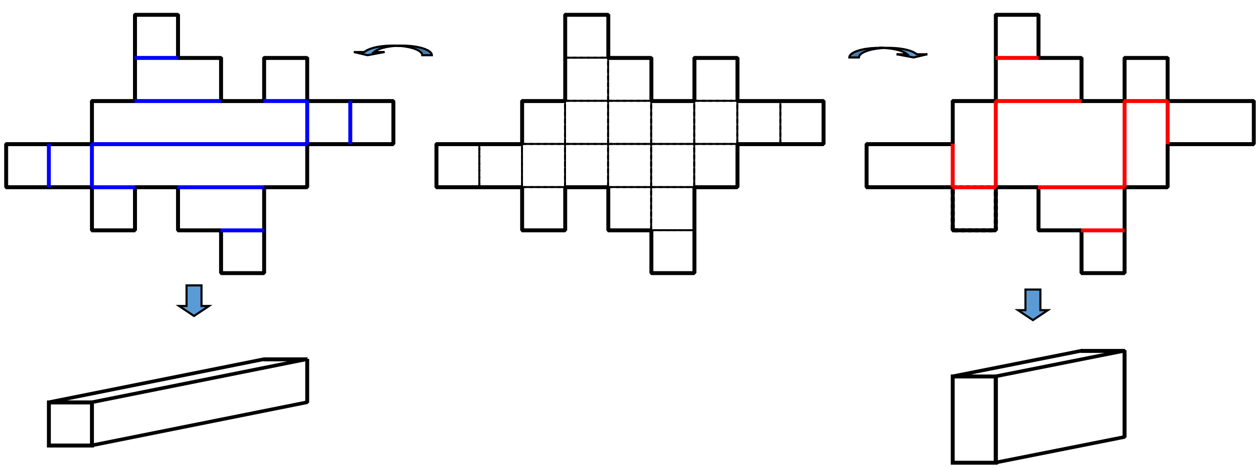
Today…

1. Basic facts for unfolding
2. <u>Polygons foldable two or more boxes</u>
3. Common unfolding of regular polyhedra (or Platonic solids)

Common Unfolding of multiple boxes

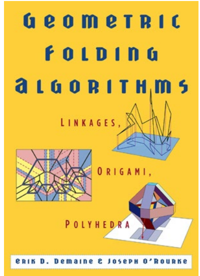
- Common unfolding of two boxes
- Common unfolding of three boxes
- And open problems….

Used as main trick in a mystery novel

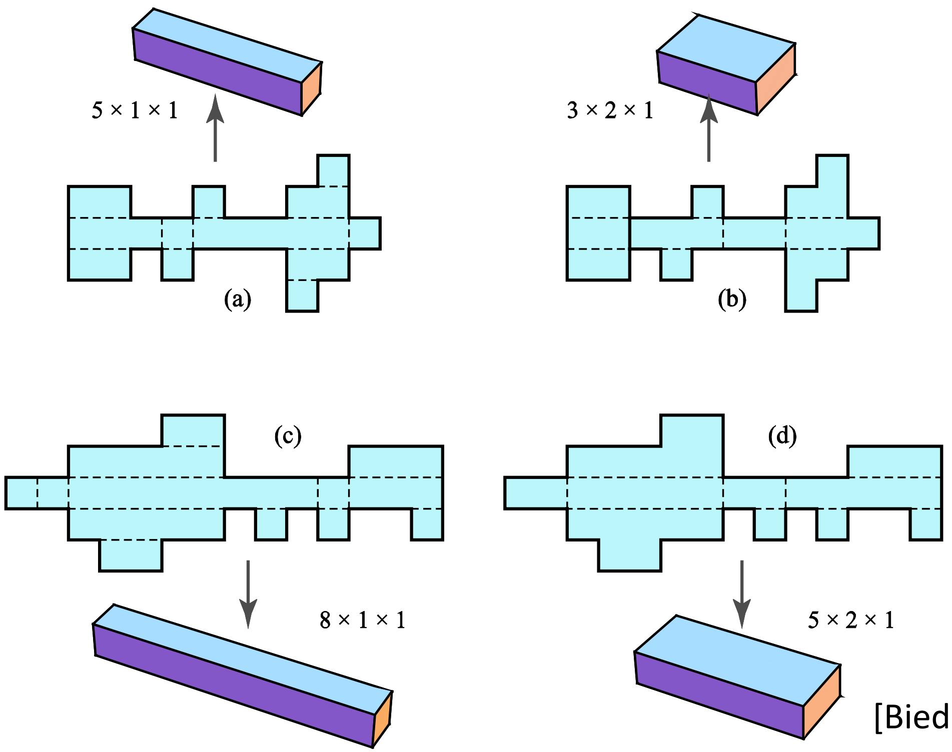


References

- Dawei Xu, Takashi Horiyama, Toshihiro Shirakawa, Ryuhei Uehara: Common Developments of Three Incongruent Boxes of Area 30, *COMPUTATIONAL GEOMETRY: Theory and Applications*, Vol. 64, pp. 1-17, August 2017.
- Toshihiro Shirakawa and Ryuhei Uehara: Common Developments of Three Incongruent Orthogonal Boxes, *International Journal of Computational Geometry and Applications*, Vol. 23, No. 1, pp. 65-71, 2013.
- Zachary Abel, Erik Demaine, Martin Demaine, Hiroaki Matsui, Guenter Rote and Ryuhei Uehara: Common Developments of Several Different Orthogonal Boxes, *Canadian Conference on Computational Geometry* (CCCG' 11), pp. 77-82, 2011/8/10-12, Toronto, Canada.
- Jun Mitani and Ryuhei Uehara: Polygons Folding to Plural Incongruent Orthogonal Boxes, *Canadian Conference on Computational Geometry* (CCCG 2008), pp. 39-42, 2008/8/13.

In  ,

- There were two unfoldings that fold to two boxes;



- Are they exceptional?
- Is there any unfolding that fold to 3 or more boxes??

[Biedl, Chan, Demaine, Demaine, Lubiw, Munro, Shallit, 1999]

Unfolding of two boxes

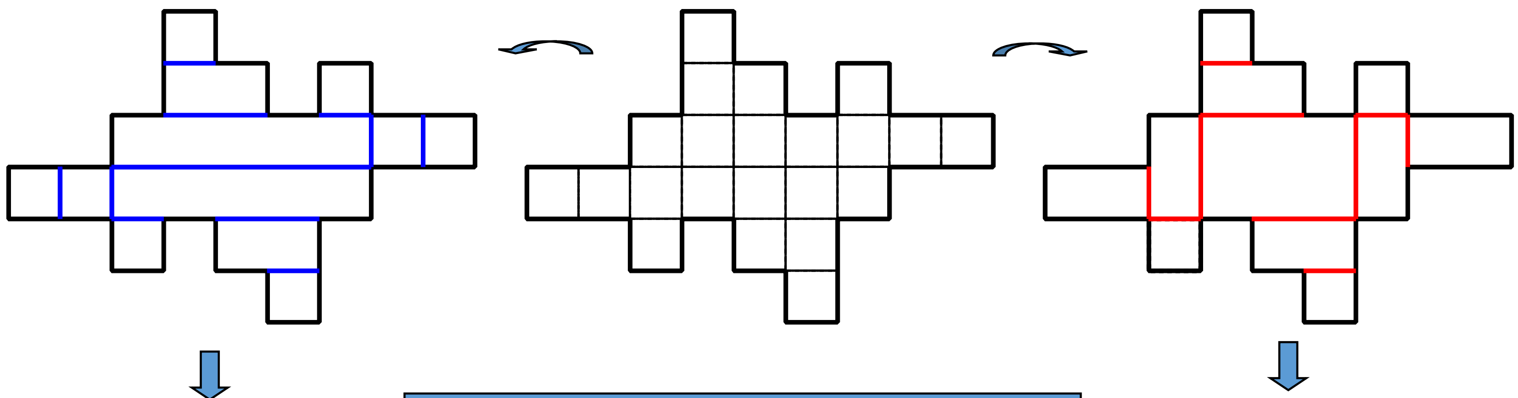
In [Uehara, Mitani 2007], randomized algorithm that looks for such polygons by *brute force*;

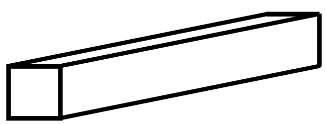
- Polygons folding into 2 boxes:
  1. There are **many (～9000)** (by supercomputer (SGI Altix 4700))
  2. Theoretically, infinitely many

Simple Observation:

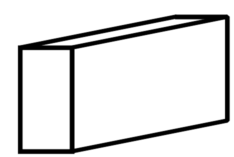
Example：
1×1+1×5+1×5
=1×2+2×3+1×3
=11（surface area: 22)

- Polygons folding to 2 different boxes





1×1×5
= a × b × c



1×2×3
= a' × b' × c'

- We fold/(cut) at an edge of **unit squares**
- Surface area: $2(ab+bc+ca)$
- Necessary condition:

$$ab+bc+ca=a'b'+b'c'+c'a'$$

It is better to have many combinations…

Simple Computation:

Surface areas;

If you try to find for three boxes,

If you try to find for four boxes,

Area	Trios	Area	Trios
<u>22</u>	(1,1,5),(1,2,3)	46	(1,1,11),(1,2,7),(1,3,5)
30	(1,1,7),(1,3,3)	70	(1,1,17),(1,2,11),(1,3,8),(1,5,5)
<u>34</u>	(1,1,8),(1,2,[illegible])	94	(1,1,23),(1,2,15),(1,3,11), (1,5,7),(3,4,5)
38	(1,1,9),(1,3[illegible]	[illegible]	(1,1,29),(1,2,19),(1,3,14), (1,4,11),(1,5,9),(2,5,7)

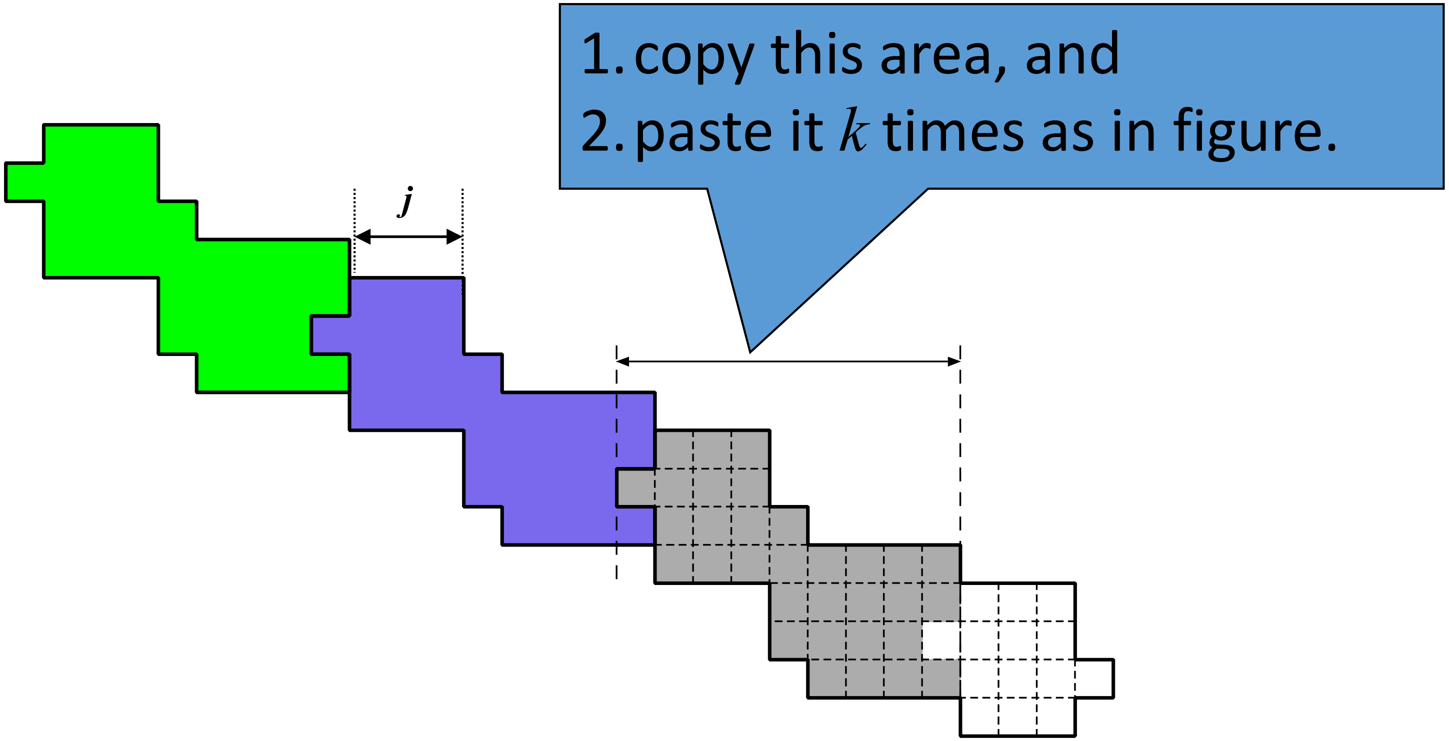
known results

My past student proved that for any *k*, there is a surface area which has *k* trios!

Unfolding of two boxes

[Theorem] There exists an infinitely many unfoldings that fold to 2 boxes.

[Proof]



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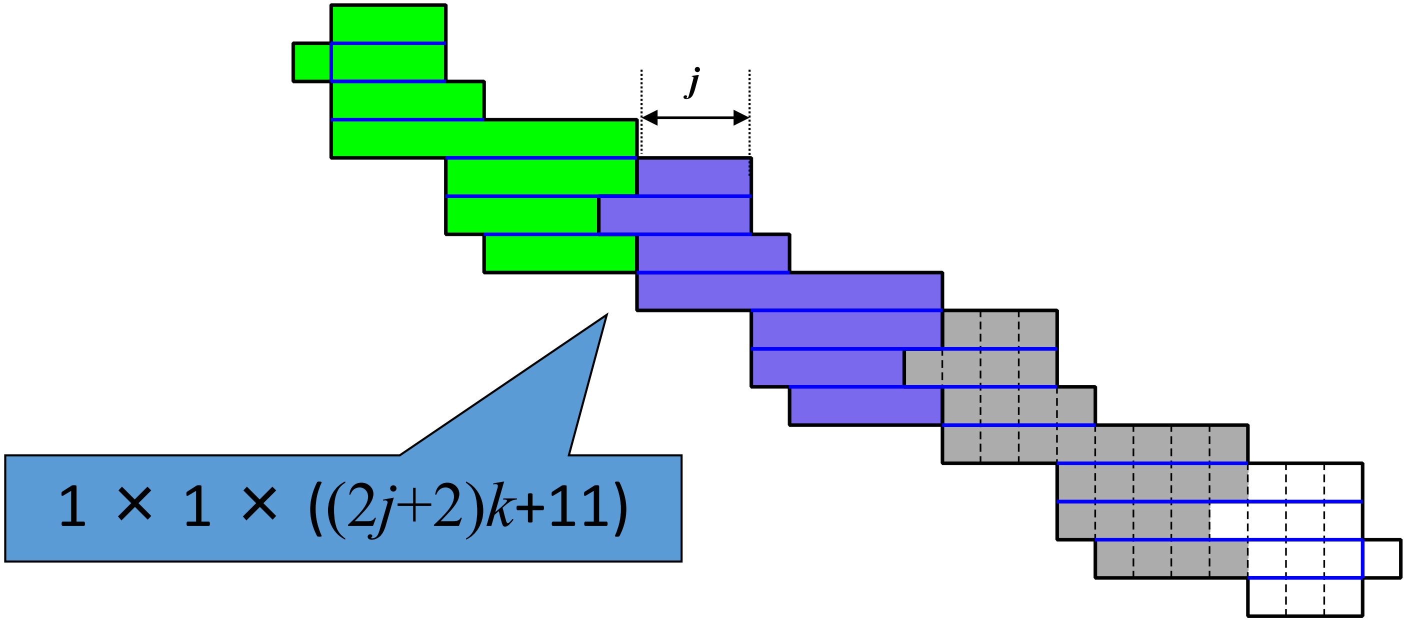

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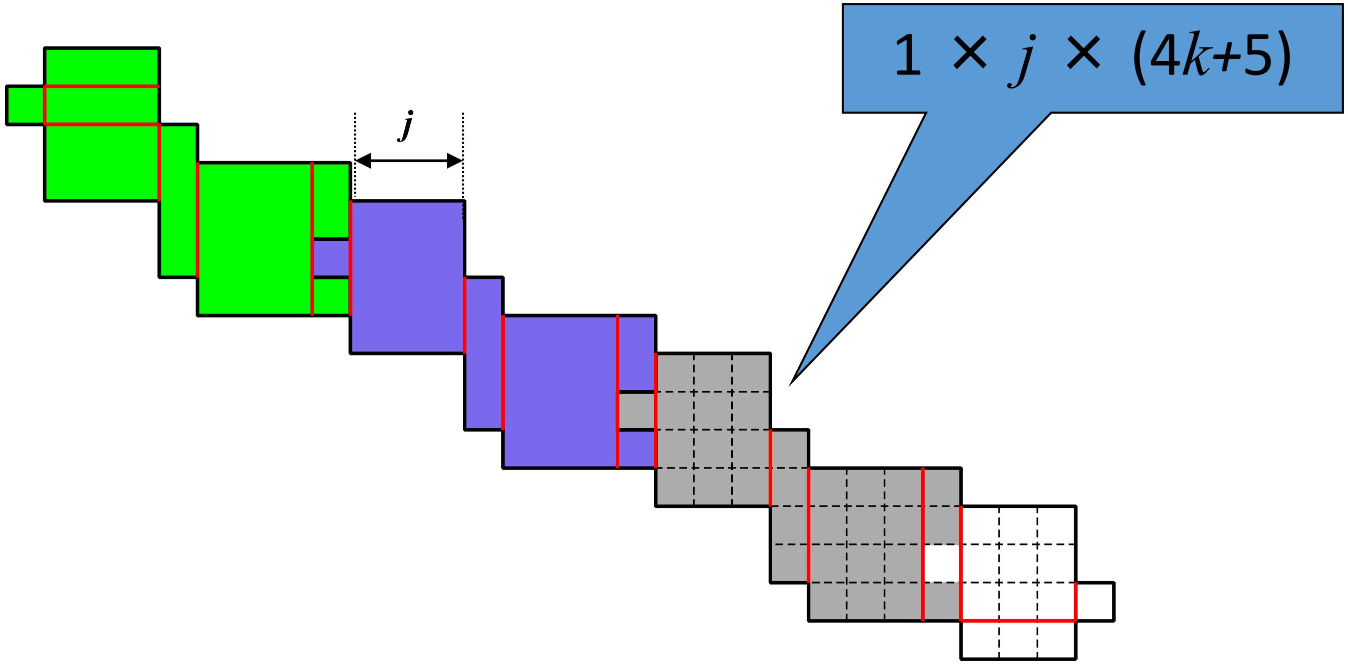
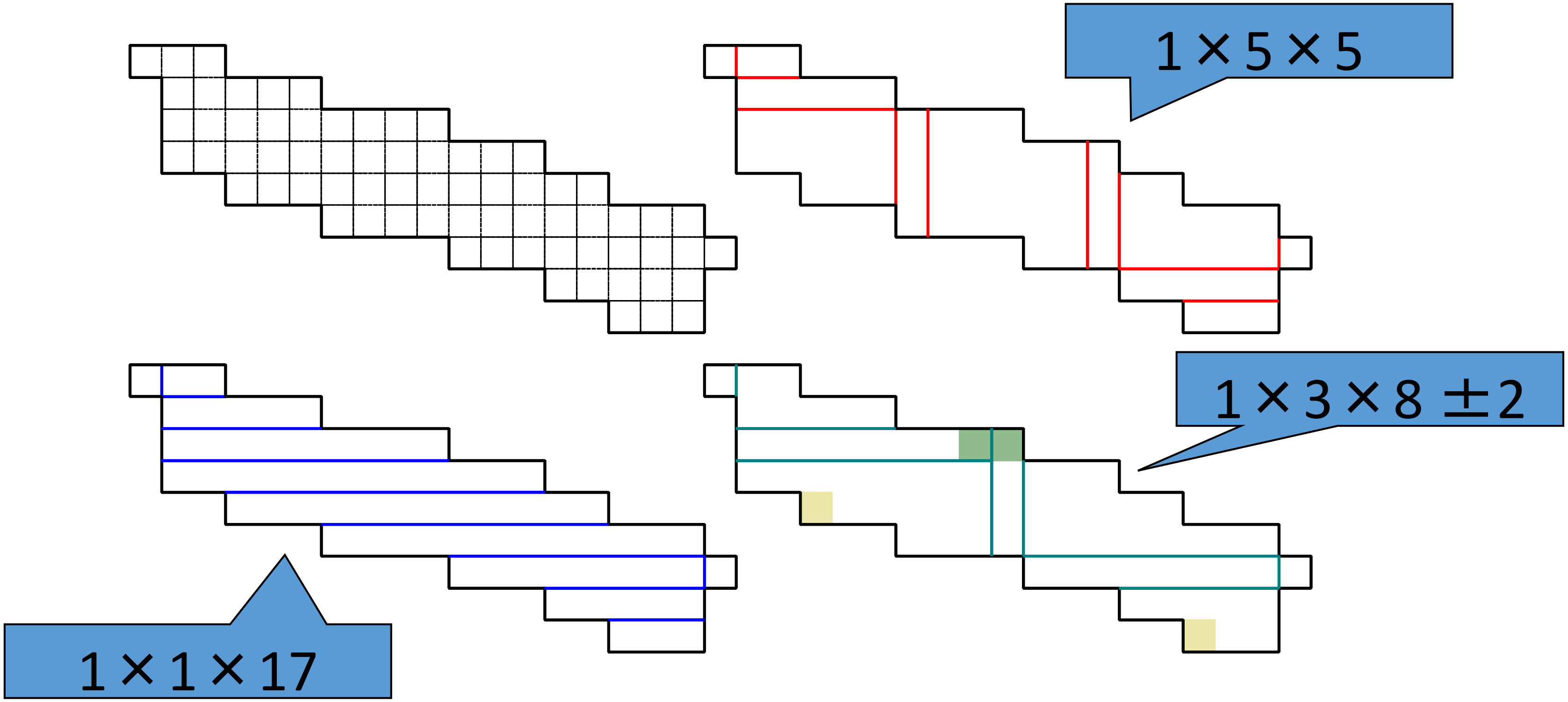


Unfolding of *three* boxes(?)

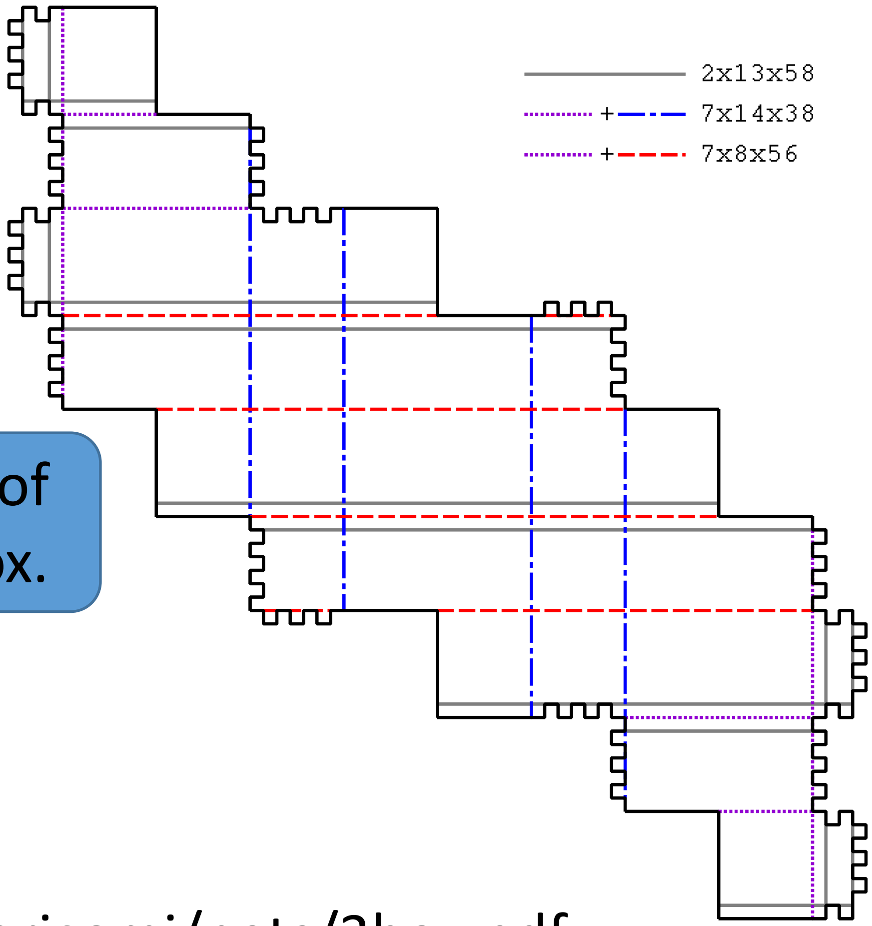
- A polygon that can fold to **three** distinct boxes…?

close one…



Unfoldings of *three* boxes (without computer!)

- In February 2012, Shirakawa (and I) finally found a polygon that folds to 3 boxes!!



[Basic Idea] From an unfolding of 2 boxes, we make **one more** box.

Available at
http://www.jaist.ac.jp/~uehara/etc/origami/nets/3box.pdf

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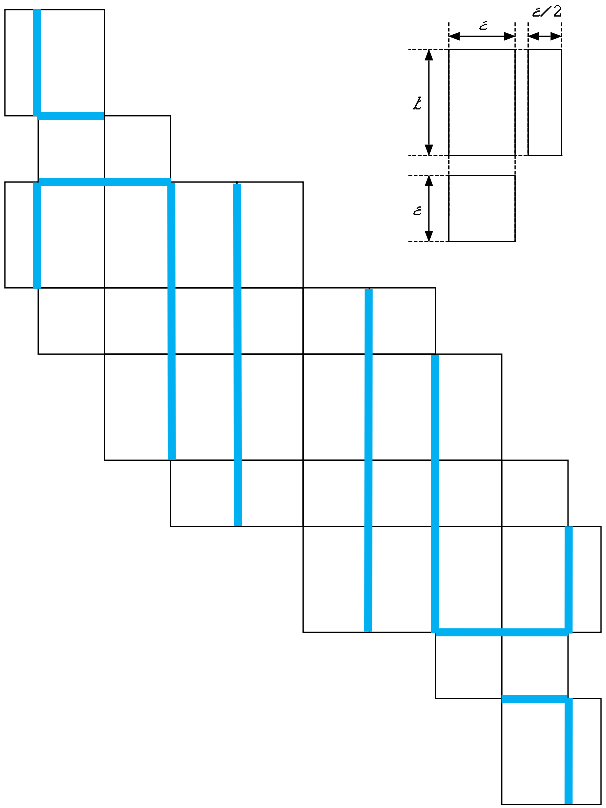

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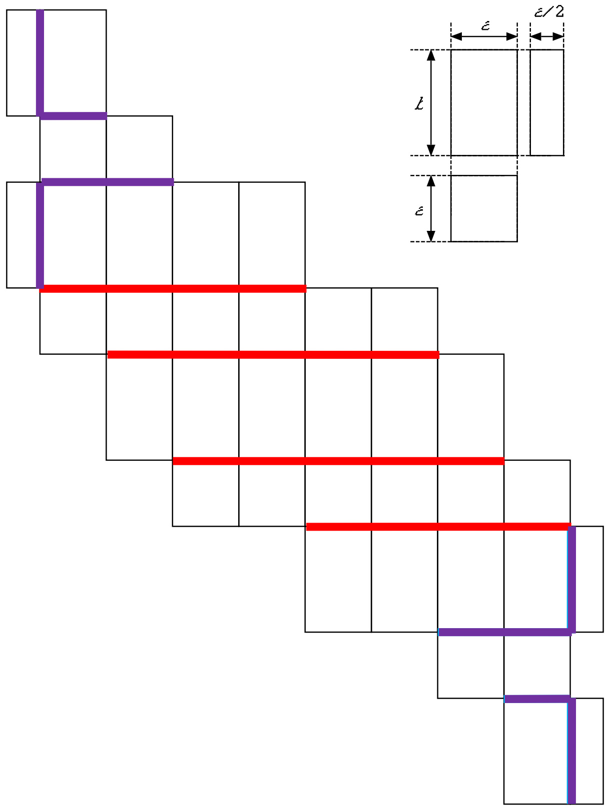

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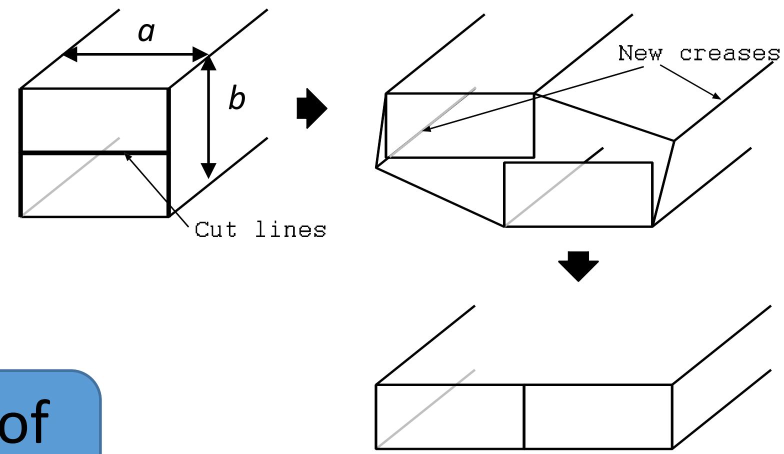

[Basic Idea] From an unfolding of 2 boxes, we make **one more** box.

One more box is obtained by this *squashing!?*

[No!!]
This works iff $a=2b$, i.e., from 1×2 rectangle to 2×1 rectangle!

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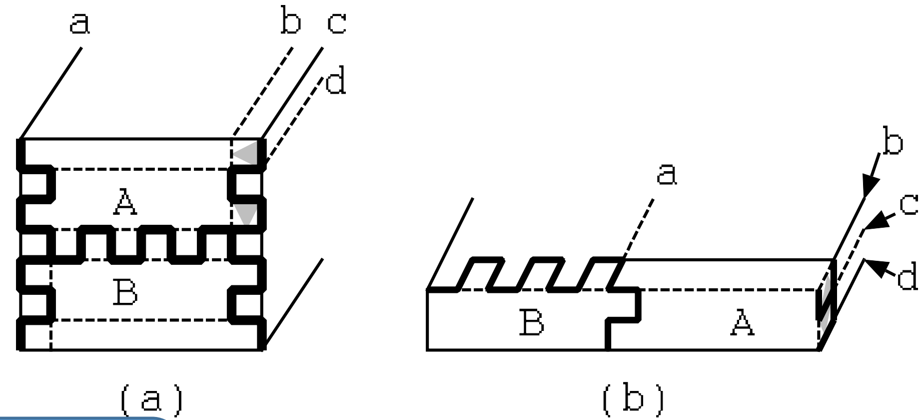

[Basic Idea] From an unfolding of 2 boxes, we make **one more** box.

[Yes… with a trick!]
This idea works; move a part of the lid to 4 *sides*!

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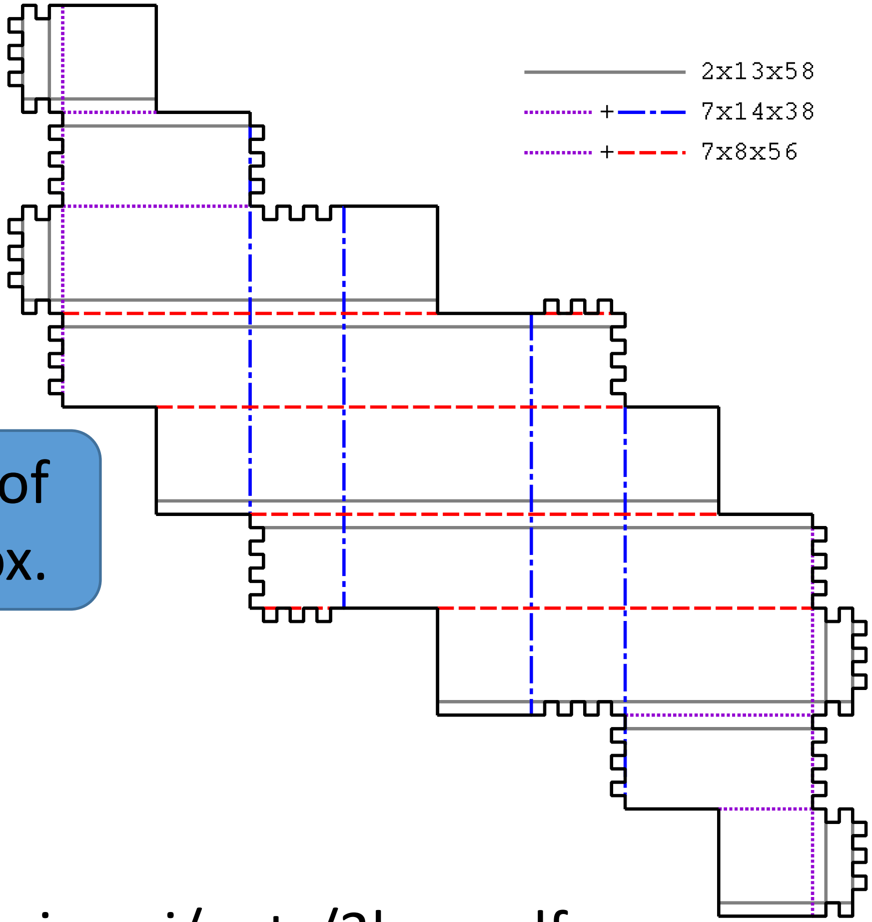

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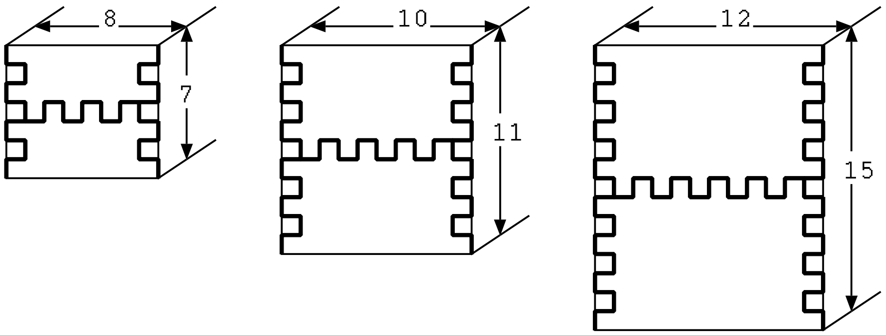


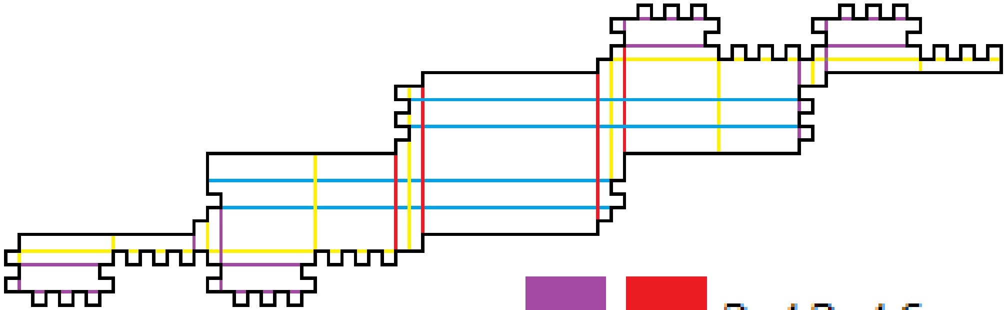
[Basic Idea] From an unfolding of 2 boxes, we make **one more** box.

[Generalization!]
- Basic box is flexible for the edge lengths.
- Zig-zag pattern can be extended.

[Theorem]
There exist an infinite number of polygons that fold into 3 different boxes.

Available at http://www.jaist.ac.jp/~uehara/etc/origami/nets/3box.pdf

Open Problems so far



- Smallest unfolding?

The current "smallest" unfolding requires

532 squares in this method.

>> the smallest area **46** that may produce

three boxes of size (1,1,11), (1,2,7), (1,3,5).

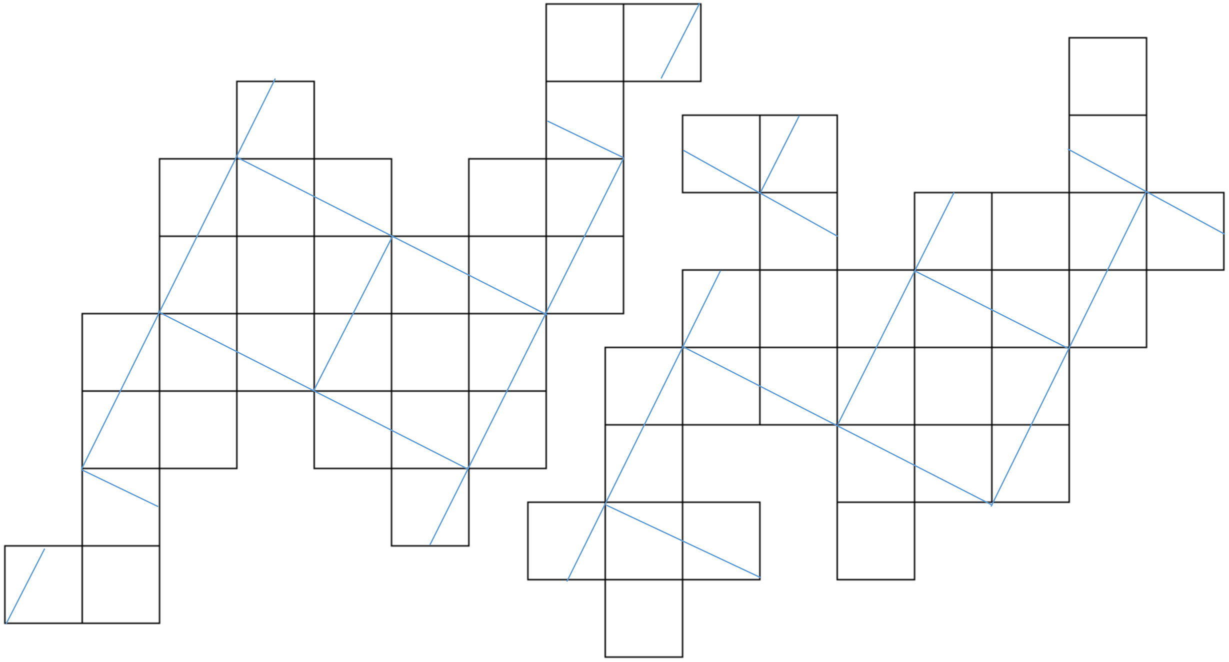
(We know: 2263 polygons of area **22** folding to (1,1,5), (1,2,3)

by 10 hours enumeration in 2011)

Is there a polygon that folds to 4 or more boxes?

Email from my puzzle friend on October 2012:

"I find unfolding of area 30 that can fold to boxes of size $1 \times 1 \times 7$ and $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$. This area allows us to fold $1 \times 3 \times 3$. So there may be a smallest polyomino that fold to three boxes if you allow to fold along diagonal."



Observation

If you try to find for three boxes,

If you try to find for four boxes,

Surface areas;

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known results

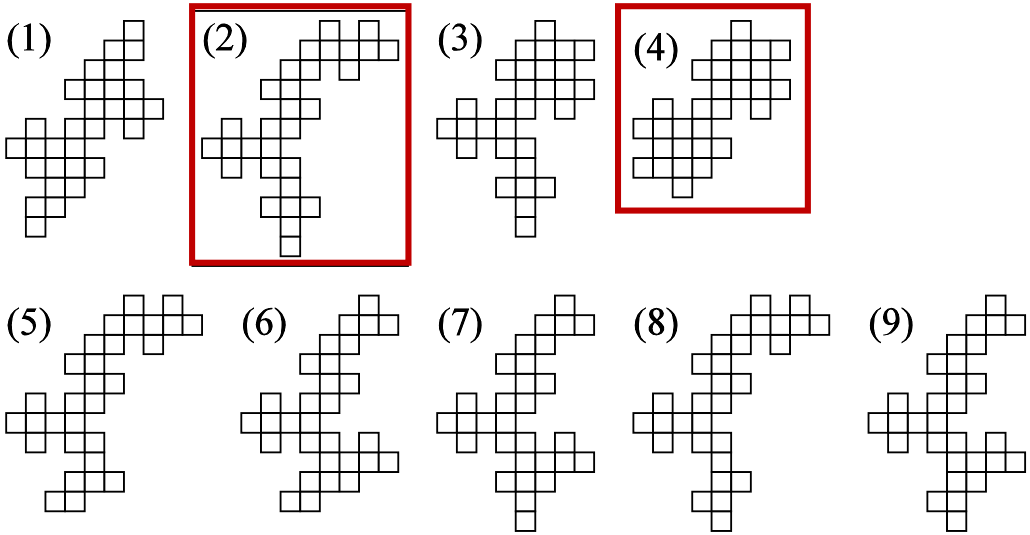
Program in <u>2011</u>:

- Enumeration of all unfolding of area 22 :
  - Two boxes of sizes 1×1×5 and 1×2×3 have 2263 common unfolding
- It run in 10 hours by a usual PC

"Area 30" sounds tractable…?

My past student succeeded! (June, 2014)

- We succeeded to enumerate all unfolding of area 30
  [Xu, Horiyama, Shirakawa, Uehara 2015]
- Summary
  - It took 2 months on a supercomputer (Cray XC 30) in JAIST.
  - We have 1080 common unfolding of two boxes of size $1 \times 1 \times 7$ and $1 \times 3 \times 3$
  - Among them, we have 9 polyominoes that fold to the third box of size $\sqrt{5} \times \sqrt{5} \times \sqrt{5}$



We had a "serendipity" (unexpected discovery): The (2) and (4) have four different ways to fold three different boxes!!

Summary

If you try to find for three boxes,

If you try to find for four boxes,

Area	Trios	Area	Trios
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38	(1,1,9),(1,3,4)	118	(1,1,29),(1,2,19),(1,3,14),(1,4,11),(1,5,9),(2,5,7)

We need more sophisticated algorithms/ideas to explore more…

Brute Force works:

- Area 22 : 10 hours on a PC in 2011
- Area 30: 2 months on a supercomputer in 2014
  - Using BDD (Binary Decision Diagram), it is improved to 10 days in 2015

Open problems

- Is there common unfolding of area 46 or 54 that can fold to three boxes?
- Is there common unfolding of 4 or more boxes?
- Is there upper bound of *k* such that “there is no common unfolding of *k* or more boxes”?
  - It is quite unlikely that one polygon can fold 10000 different boxes…?

Recent work and future work

- More general problem:

  For a given polygon P and a convex polyhedron Q, determine if P can fold to Q or not.

Known/related results:

1. There is a general pseudo-polynomial time algorithm for general polygon P and convex polyhedron Q, but…
   - The algorithm runs in $O(n^{456.5})$ time! (Kane, et al, 2009)
2. We solved if Q is "some box"; (size is not given)
   - Koichi Mizunashi, Takashi Horiyama, and Ryuhei Uehara: Efficient Algorithm for Box Folding, Journal of Graph Algorithms and Applications, accepted, 2019.

There are many unsettled problems between them!

Computational ORIGAMI=

Geometry＋Algorithm＋Computation

- Mathematics
- Theoretical Computer Science
- Real High Performance Computing

- Many Applications from micro-size to space-size
  - Bioinfomatics (e.g., DNA folding),
  - Robotics, packaging,
  - Architecture

Let's join it!

- Many young researchers;
  - even undergrad students, highschool students!