Polygons Folding to Plural Incongruent Orthogonal Boxes

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Abstract

We investigate the problem of finding orthogonal polygons that fold to plural incongruent orthogonal boxes. There are two known polygons that fold to produce two incongruent orthogonal boxes. In this paper, we show that there are infinite such polygons. We also show that there exists a tile that produces two incongruent orthogonal boxes.

1 Introduction

Polygons that can fold to a convex polyhedron have been investigated since Lubiw and O'Rourke posed the problem in 1996 [4]. Recently, Demaine and O'Rourke published a book about geometric folding algorithms that includes many results about such polygons [2, Chapter 25]. One of the many interesting problems in this area is that whether there exists a polygon that folds to plural incongruent orthogonal boxes. Biedl et al. answered "yes" by finding two polygons that fold to two incongruent orthogonal boxes [1] (see also [2, Figure 25.53]). However, are these two polygons exceptional? We show that the answer is "no." In this paper, we first report that there are more than two thousands such polygons of several sizes. These polygons were found by a randomized algorithm that repeatedly produces many nets of orthogonal boxes at random, and matches them in a huge hash table. Some of those polygons can be extended to general size. Using this fact, we also show that there exist an infinite number of polygons that can fold to two orthogonal boxes. Moreover, we show that there exists a simple polygon that can fold to two orthogonal boxes, and that tiles the plane. This pattern may be used to produce two kinds of boxes of two different volumes on demand without loss of material.

2 Preliminaries

In this paper, we concentrate on orthogonal polygons that consist of unit squares. For a positive integer S, we denote by $P(S)$ the set of three integers a, b, c with $0 < a \le b \le c$ and $ab + bc + ca = S$, i.e., $P(S) = \{(a, b, c) \mid ab + bc + ca = S\}$. Clearly, it is necessary to satisfy $|P(S)| \ge k$ to have a polygon of size $2S$ that can fold to k incongruent orthogonal boxes. For example, the two known polygons in [1] correspond to $P(11) = \{(1, 1, 5), (1, 2, 3)\}$ and $P(17) = \{(1, 1, 8), (1, 2, 5)\}$. Similarly, we have $P(15) = \{(1, 1, 7), (1, 3, 3)\}$, $P(23) = \{(1, 1, 11), (1, 2, 7), (1, 3, 5)\}$, $P(35) = \{(1, 1, 17), (1, 2, 11), (1, 3, 8), (1, 5, 5)\}$, $P(47) = \{(1, 1, 23), (1, 2, 15), (1, 3, 11), (1, 5, 7), (3, 4, 5)\}$, $P(59) = \{(1, 1, 29), (1, 2, 19), (1, 3, 14), (1, 4, 11), (1, 5, 9), (2, 5, 7)\}$, and so on.

Let B be an orthogonal box of size $a \times b \times c$. Then there are six faces that consist of two rectangles of size $a \times b$, $b \times c$, and $c \times a$, respectively. We regard each rectangle as a set of unit squares. That is, B consists of $2(ab + bc + ca)$ unit squares. Then, for B, we define a dual graph $G(B) = (V, E)$ of B as follows; V is the set of $2(ab + bc + ca)$ unit squares, and E contains an edge $\{u, v\}$ iff two unit squares u and v share an edge on B, or they are incident on B. It is easy to see that $G(B)$ is a 4-regular graph of $2(ab+bc+ca)$ vertices, and hence $|E| = 4(ab+bc+ca)$. Then we have the following observation:

Observation 1 *Let T be a spanning tree of $G(B)$ for some B. For every edge $\{u, v\}$ not in T, we cut the edge shared by two unit squares u and v on B. Then, we obtain a net P of B.*

That is, we can make a net P of B for any orthogonal box B. In the case, we say that the spanning tree T *produces* P. However, spanning trees themselves are not good to represent nets of a box. Suppose that a polygon P can fold to an orthogonal box B. In general, P contains a rectangle R of size $a \times b$ with $a > 1$ and $b > 1$. Then, no spanning tree T generates P since T forces unnecessary cuts of inside of R. The following lemma patches this problem.

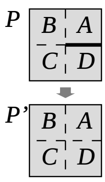

Figure 1: Gluing.

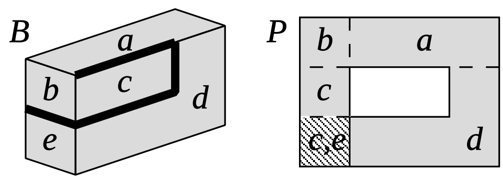

Figure 2: A half of a nonsimple polygon folding to a box.

Lemma 1 *Let P be a polygon that can fold to a box B. If P has a cut between two unit squares A and D*

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in Figure 1, we glue them and obtain P'. Then P' also can fold to B.

Proof. Since B is a convex orthogonal box, it follows. □

Repeating the gluing in Lemma 1, we obtain a polygon P that has no two consecutive identical edges, which means P contains no unnecessary cuts. From the viewpoint of programming, it is sufficient to represent each polygon P by a usual 0/1 matrix in a natural way, and ignore such cuts. One may think that any polygon that can fold to a box is simple. However, it is not the case.

Lemma 2 *Let B be an orthogonal box and P a polygon that can fold to B. Then, P is not necessarily simple.*

Proof. For B of size $1 \times 2 \times 3$, we make a (half of) polygon P as in Figure 2. Then, clearly, P is a polygon that can fold to B, but P is not simple. □

3 Algorithm

Our algorithm is a quite simple randomized one described below:

Input : S with $|P(S)| > 1$;
Output: Polygons of size $2S$ that fold to plural boxes;

clear a hash table H;
while *true* **do**
  choose a type $t = (a, b, c)$ in $P(S)$ at random;
  generate a spanning tree T of $G(B)$ for an orthogonal box B of size $a \times b \times c$ at random;
  represent a polygon P corresponding to T by a 0/1 matrix;
  if (t', P) *is in* H *with* $t \neq t'$ **then** output P (and all associate types);
  if P *is not in* H **then** add (t, P) into H;
end

We aim at finding polygons shared by two or more types. Hence, the algorithm ignores weak points mentioned in Preliminaries. More precisely, the algorithm has the following flaws; (1) it does not generate the polygons *uniformly* at random, (2) some polygons overlap (by Lemma 2). Moreover, even if the polygon P does not overlap, two nonincident squares on the box B can share a common edge on P (by Lemma 2; if we have a cut between d and e in Figure 2, a, b, c, and d make a hole in P). Since the information is not represented on a 0/1 matrix, (3) some polygons P contain holes and have to be cut in differently to produce two distinct boxes.

Fortunately, the flaws cause few errors through our experiments; in fact, among 2165 outputs, the algorithm produced 2139 simple polygons, which are solutions, and only 26 non-simple polygons, which are not solutions. We note that from the algorithmic point of view, it is easy to check (2) and (3) in linear time when the algorithm outputs each solution.

Table 1: Experimental results (1)

$2S(S)$	$\|P(S)\|$	∼RG($\times 10^7$)	Sols	Errs
22(11)	2	6.7	541	15
30(15)	2	18.6	72	1
34(17)	2	28.4	708	0
38(19)	2	30.4	41	0
46(23)	3	191.0	660	8
54(27)	3	126.7	3	0
58(29)	3	89.3	37	0
62(31)	3	82.4	5	0
64(32)	3	204.8	56	2
70(35)	4	91.3	14	0
88(44)	4	217.0	2	0
94(47)	5	51.3	0	0
118(59)	6	35.5	0	0
Total	-	-	2139	26

Table 2: Experimental results (2)

$2S(S)$	Types	Sols	Errs
46(23)	(1,1,11), (1,3,5)	568	3
	(1,2,7), (1,3,5)	92	5
54(27)	(1,1,13), (3,3,3)	2	0
	(1,3,6), (3,3,3)	1	0
58(29)	(1,1,14), (1,4,5)	37	0
62(31)	(1,3,7), (2,3,5)	5	0
64(32)	(1,2,10), (2,2,7)	50	2
	(2,2,7), (2,4,4)	6	0
70(35)	(1,1,17), (1,5,5)	3	0
	(1,2,11), (1,3,8)	11	0
88(44)	(2,2,10), (1,4,8)	2	0

4 Experimental results

We first ran the algorithm on a laptop (IBM ThinkPad X40: 1 Processor with 1.5GB Memory). This generated approximately 3×10^6 polygons in 1 hour, and obtained around 100 solutions for $P(11)$. To experiment more efficiently, we used a supercomputer (SGI Altix 4700: 96 Processors with 2305GB Memory). We used an implement of the Mersenne Twister algorithm[1] to generate random numbers. Our results are summarized in Tables 1 and 2. In Table 1, "$2S(S)$" denotes the (half) size of a polygon, "$|P(S)|$" denotes the number of distinct box types, "RG" denotes the number of random generations, "Sols" denotes the number of simple polygons that can fold to two incongruent orthogonal boxes, and "Errs" denotes the number of non simple polygons. For example, for $P(11)$, the algorithm generates around 6.7×10^7 nets of boxes of size $(1, 1, 5)$ *or* $(1, 2, 3)$, and we have 556 outputs. Among them, 15 polygons have a hole, and hence we have 541 distinct simple polygons that can fold

[1] `http://www.math.sci.hiroshima-u.ac.jp/~m-mat/MT/emt.html`

to boxes of size $(1, 1, 5)$ *and* $(1, 2, 3)$. In total, we have 2139 distinct simple polygons that can fold to two incongruent orthogonal boxes. For each S with $|P(S)| > 2$, more details can be found in Table 2. All cases are checked in parallel on the machine, and the computations take from a few days to a few weeks (we stopped execution when each process requires too much memory). Some solutions are illustrated in Figure 7, and all solutions can be found at `http://www.jaist.ac.jp/~uehara/etc/origami/nets/index-e.html`. After these experiments, we still have no polygon that can fold to three (or more) incongruent orthogonal boxes. We note that some values of S are related; for example, the solutions for $P(11)$ give the solutions for $P(44)$ by dividing a unit square into four unit squares. Although we have 541 solutions for $P(11)$ after 6.7×10^7 random generations (it takes 3 days), we have only two solutions for $P(44)$ after 217.0×10^7 random generations (it takes 1 month). These two solutions for $P(44)$ do not correspond to any solution for $P(11)$. Some special polygons found in the solutions are below.

Tiling The discovered polygonal patterns reminded us of *tilings*. Indeed, there exists a simple polygon that can fold to two incongruent orthogonal boxes and it forms a tiling. The polygon in Figure 3 can fold to two boxes of size $1 \times 1 \times 8$ and $1 \times 2 \times 5$, and it tiles the plane.

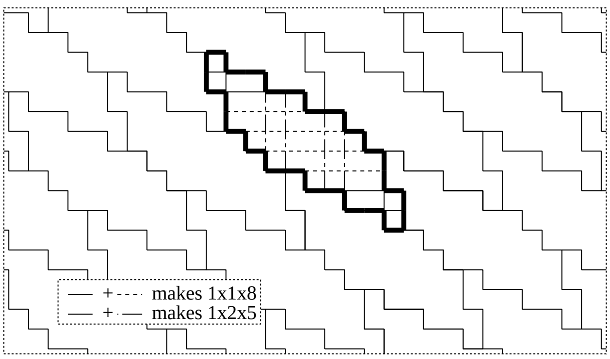

Figure 3: Polygon folding to two boxes of $1 \times 1 \times 8$ and $1 \times 2 \times 5$, and tiling the plane.

We note that the boxes with the common polygon form "*double packable solids*" introduced by Akiyama [3, Section 3.5.2]. Moreover, we can make two kinds of the boxes of volumes 8 and 10 on demand!

Disjoint crease patterns There exists a simple polygon that can fold to two incongruent orthogonal boxes and that foldings to two boxes are disjoint; the last polygon in Figure 7 satisfies the property.

Cross-free patterns There exists a simple polygon that can fold to two incongruent orthogonal boxes and that foldings to two boxes are cross free. The second last polygon in Figure 7 satisfies the property. We note that the previously known results in [1] also satisfy the property.

We have not checked if there exists a simple polygon such that foldings are disjoint and cross free.

5 Infinite polygons

A natural question is whether or not there are infinite distinct[2] polygons that can fold to plural boxes? The answer is "yes." Some polygons obtained by the experiments can be generalized. From two of them, we have the following theorem.

Theorem 3 *For any positive integer k, there is a distinct polygon that can fold to two incongruent orthogonal boxes of sizes (1)* $1 \times 1 \times (6k + 2)$ *and* $1 \times 5 \times 2k$, *and (2)* $1 \times 1 \times (8k + 11)$ *and* $1 \times 3 \times (4k + 5)$.

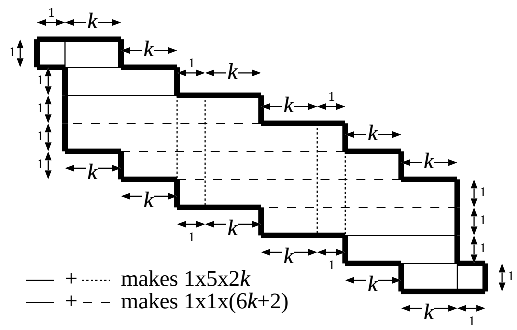

Figure 4: Polygon folding to two boxes of $1 \times 1 \times (6k+2)$ and $1 \times 5 \times 2k$ by stretch.

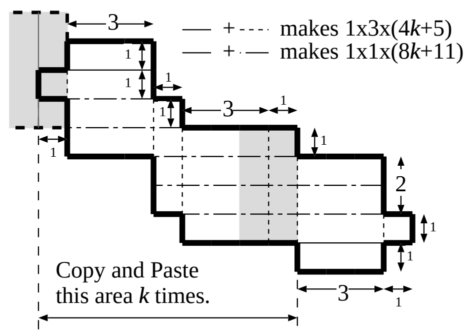

Figure 5: Polygon folding to two boxes of $1 \times 1 \times (8k+11)$ and $1 \times 3 \times (4k + 5)$ by spiral.

[2] Precisely, *distinct* means $gcd(a, b, c, a', b', c') = 1$ for two boxes of size $a \times b \times c$ and $a' \times b' \times c'$.

Proof. The first one is obtained by stretching a polygon. For any positive integer k, Figure 4 gives a polygon that satisfies (1). The second one is obtained by a spiral extension of a polygon. For any positive integer k, we copy in the leftside polygon in Figure 5 and glue it to the leftmost square (with overlapping at gray areas) and repeat it k times. Then the polygon satisfies (2). □

Corollary 4 *There exist an infinite of distinct polygons that can fold to two incongruent orthogonal boxes.*

6 Concluding remarks

From the theoretical point of view, uniform random generation and enumeration of all simple polygons for a given box are interesting problems. However, those algorithms are not necessarily useful to find polygons that can fold to plural incongruent orthogonal boxes. Indeed we search "similar" polygons heuristically to find such polygons.

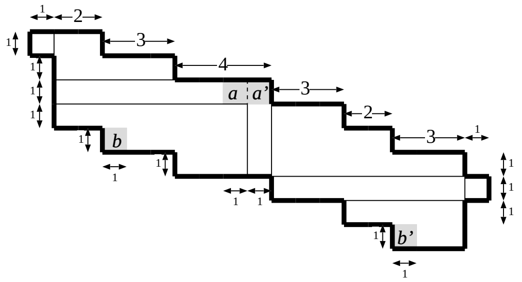

Figure 6: A polygon folding to two boxes of $1 \times 1 \times 17$, $1 \times 5 \times 5$, and "close" to $1 \times 3 \times 8$.

It is an open question if a polygon exists that can fold to three or more orthogonal boxes. The author conjectures "yes;" through experience, there is a polygon that seems to be "close" to the answer. The polygon in Figure 6 can fold to two boxes of size $1 \times 1 \times 17$ and $1 \times 5 \times 5$ in the similar ways in Figure 4. Moreover, it also can fold to the box of size $1 \times 3 \times 8$ with only two overlapping squares (and hence with two holes); a and b overlap with a' and b', respectively (with a cut between a and a').

Acknowledgements

The author thanks Joseph O'Rourke, who gave an interesting talk at JAIST, which involved the author in this work. The author also thank to Erik Demaine, who kindly sent the note [1] to the author.

References

[1] T. Biedl, T. Chan, E. Demaine, M. Demaine, A. Lubiw, J. I. Munro, and J. Shallit. Notes from the University of Waterloo Algorithmic Problem Session. Sept. 8 1999.

[2] E. D. Demaine and J. O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, 2007.

[3] M. Kano, M.-J. P. Ruiz, and J. Urrutia. Jin Akiyama: A Friend and His Mathematics. *Graphs and Combinatorics*, 23[Suppl]:1–39, 2007.

[4] A. Lubiw and J. O'Rourke. When Can a Polygon Fold to a Polytope? Technical Report Technical Report 048, Department of Computer Science, Smith College, 1996.

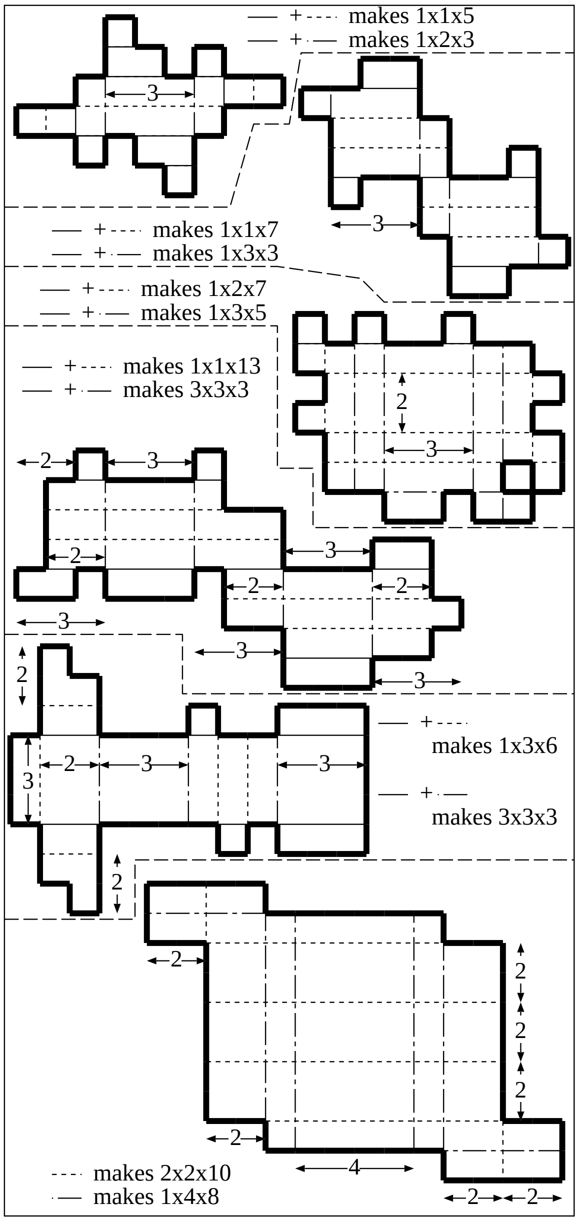

Figure 7: A part of solutions.