Another Measure for the Lexicographically First Maximal Subgraph Problems and Its Threshold Value on a Random Graph*

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Abstract

We investigate structural parameters of a graph that express the boundary of the parallel complexity of the lexicographically first maximal independent set (LFMIS) problem. The longest directed path length (LDPL), which is introduced in the paper, is a better measure than previously known one. The parallel complexity of the LFMIS problem on a graph gradually increases as the value measured on the graph grows: The LFMIS problem on a graph of LDPL $O(\log^k n)$ is in NC^{k+1}, and the problem on a graph of LDPL $\Theta(n^\epsilon)$ is P-complete. We also investigate the limit of the measure. We construct a kind of the lexicographically first maximal subgraph problems such that the problem is P-complete even if the LDPL of the input graph is restricted to 1. Finally we discuss the probability that a random graph has LDPL l and show that its threshold value is $\frac{l}{n}$. This implies that a random graph of which each edge exists with probability p has LDPL $\Theta(np)$ with high probability. The result also show the limit of the LDPL on the average case.
Keywords: Analysis of algorithms, NC algorithms, P-completeness, the lexicographically first maximal subgraph problems, threshold function of a random graph.

1 Introduction

The parallel complexity of the problems to find a maximal vertex-induced subgraph that satisfies a specified graph property has been widely investigated. Karp and Wigderson first showed that the typical maximality problem, the maximal independent set (MIS) problem, is in the class NC [1]. Since then, much work has been devoted to the study of parallel complexity of maximality problems [2, 3, 4, 5, 6, 7, 8]. On the other hand, the lexicographically first maximal independent set (LFMIS) problem is a typical P-complete problem [9], and P-completeness of the lexicographically first maximal subgraph (LFMS) problems for some graph properties π has been shown [10, 7] (see also [11] for a comprehensive reference).

One of the approaches to make clear the boundary between the classes NC and P is to find (sufficient) conditions for problems to be in NC or to be P-complete. Uehara provides a "measure" of sequentiality for the LFMIS problem [12]. The measure is the *maximum directed tree size* (MDTS) of G, which is defined by the maximum number of vertices of directed trees in the directed acyclic graph of G. He showed that the parallel complexity of the LFMIS problem on a graph gradually increases as the value measured on the graph grows: The LFMIS problem on a graph of MDTS $O(\log^k n)$ is in NC^{k+1}, and the problem on a graph of MDTS $\Theta(n^\epsilon)$ is P-complete. We introduce another measure, the *longest directed path length* (LDPL) of G, which is defined by the length of the longest directed path on the directed acyclic graph of G. We first show that the LDPL is a better measure than the MDTS.

We next show the limit of the measures. In [12], Uehara claimed that the MDTS also measures the sequentiality of the LFMS problems for a graph property π such that π is nontrivial, hereditary on vertex-induced subgraphs, satisfied by all independent edges, and NC testable. However, his proof is

*This research was partially supported by the Ministry of Education, Science, Sports and Culture, Grant-in-Aid for Encouragement of Young Scientists, 10780203, 1998.

incomplete. As a counterexample, we reduce the LFMIS problem to a kind of the LFMS problems. This reduction shows that there exists an LFMS problem that is P-complete even if the MDTS of input graph is restricted to 3. This also implies that the limit of the LDPL: The problem is P-complete even if the LDPL of input graph is restricted to 1.

Finally we also show the limit of the LDPL on the average case. We consider the average value of the LDPL on a random graph. (For the detail of a random graph, see [13, 14, 15] for example.) The property of LDPL l seems to be similar to the graph property that contains a path of length l (see, e.g. [15, Figure 13.1]). However, interestingly, these properties are completely different, and so their threshold values are. Let $Q(l)$ be the graph property that the LDPL of the graph is at least l. For a random graph $G(n,p)$, the threshold value for the property $Q(l)$ is $\frac{l}{n}$ when $\lim_{n\to\infty} l = \infty$. (Cf. The threshold value of the graph property that a graph contains a path of length l is $n^{-(l+1)/l}$ [15, Figure 13.1].) The threshold value implies that the LDPL of a random graph $G(n,p)$ is $\Theta(np)$ with high probability. Thus, for any fixed positive real number $0 < p < 1$, the LDPL of a random graph $G(n,p)$ is $\Theta(n)$ with high probability. On the other hand, the LFMIS problem on the random graph $G(n,p)$ can be solved in $O(\log n)$ time (on average) in parallel [16, 17]. That is, there seems to exist a gap between the measure LDPL and the time required to solve the LFMIS on the random graph.

2 Preliminaries

We will deal only with graphs and digraphs without loops or multiple edges. Throughout the paper, unless stated otherwise, G always denotes the input (undirected) graph, $V = \{0, 1, \cdots, n-1\}$ and E denote the set of vertices and edges in G, respectively. Without loss of generality, we assume that G is connected. The *neighborhood* of a vertex v in G, denoted by $N_G(v)$, is the set of vertices in G adjacent to v. The *degree* of a vertex v in G is $|N_G(v)|$, and denoted by $\mathrm{d}_G(v)$. Vertices of degree 0 are called *isolated vertices*. For $U \subseteq V$, $N_G(U)$ is $\cup_{u\in U} N_G(u)$, and $G[U]$ is the graph (U,F), where $F = \{\{u,v\} \mid u,v \in U \text{ and } \{u,v\} \in E\}$. For a graph G, $|G|$ denotes the number of vertices in G.

Let $X = \{x_1, x_2, \cdots, x_k\}$ and $Y = \{y_1, y_2, \cdots, y_h\}$ be any subsets of V. (We assume that sets are always sorted, that is, $x_i < x_j$ and $y_i < y_j$ for each $1 \le i < j \le k, h$.) Let $<$ be the total ordering on the sets X and Y defined as follows: $X < Y$ if and only if $x_i = y_i$ for every i with $1 \le i \le k$, or there is an index $i' \ge 1$ such that $x_j = y_j$ for every $1 \le j < i'$, and $x_{i'} < y_{i'}$.

A subset U of V is called an *independent set* if $G[U]$ only contains isolated vertices. A *maximal independent set* (MIS) in G is an independent set that is not properly contained in any other independent set. The MIS problem is to find, given a graph G, an MIS in G. The *lexicographically first maximal independent set* (LFMIS) in G is the MIS I in G such that $I < J$ for every MIS J in G. The LFMIS problem is to find, given a graph G, the LFMIS in G. The *distance* of two vertices in G is the length of a shortest path joining the vertices. The *diameter* of a graph is the greatest distance between any two vertices in the graph.

For $G = (V,E)$, $\vec{G} = (V,\vec{E})$ is defined by the directed acyclic graph (DAG) obtained from G by replacing each edge $\{u,v\}$ by the arc $(\min\{u,v\}, \max\{u,v\})$. (It is easy to see that the resulting graph $\vec{G}$ is acyclic for any graph G.) The *neighborhood* of a vertex v in $\vec{G}$, denoted $N_{\vec{G}}(v)$, is the set of vertices u in $\vec{G}$ such that $(v,u) \in \vec{E}$. The *outdegree* of a vertex v in $\vec{G}$, denoted by $\mathrm{d}^+_{\vec{G}}(v)$, is $|N_{\vec{G}}(v)|$, and the *indegree* of a vertex v in $\vec{G}$, denoted by $\mathrm{d}^-_{\vec{G}}(v)$, is $|N_G(v)| - |N_{\vec{G}}(v)|$. We mention that any DAG $\vec{G}$ has at least one vertex v with $\mathrm{d}^-_{\vec{G}}(v) = 0$, and at least one vertex u with $\mathrm{d}^+_{\vec{G}}(u) = 0$ [18, Theorems 16.2 and 16.2']. Especially, $\mathrm{d}^-_{\vec{G}}(0) = 0$ and $\mathrm{d}^+_{\vec{G}}(n-1) = 0$ for any DAG $\vec{G}$.

Let n be a positive integer, and p be a positive real number with $0 < p < 1$ (and p may depend on n). The random graph $G(n,p)$ is a probability space over the set of graphs on the vertex set $\{0, 1, \cdots, n-1\}$ determined by $\Pr[\{i,j\} \text{ is an edge in } G] = p$, with these events mutually independent. (For the detail of a random graph, see [13, 14, 15] for example.) Let Q be a graph property that is not destroyed by the addition of edges to a graph. A function $r(n)$ is called a *threshold function for* Q if $\lim_{n\to\infty} p(n)/r(n) = 0$ implies that almost no graph has property Q, and $\lim_{n\to\infty} p(n)/r(n) = \infty$ implies that almost every graph has property Q.

We use P to denote the class of all polynomial time computable problems, NL to denote the class of all decision problems solvable by nondeterministic Turing machines (NTMs) that use space bounded by $O(\log n)$, and NC^k to denote the class of all problems computable by a uniform polynomial size circuit family of depth $O(\log^k n)$, where n is the size of input, and k is some positive constant. The class NC is defined by $\cup NC^k$ (for further details, refer to [9, 19]). Although the P-completeness is defined via NC^1-reducibility in [9], we use the log space reducibility simply as in [10]: A problem F_0 is said to be *P-complete* if F_0 is in P and for each F in P there are log space computable functions f and g such that $F(x) = g(F_0(f(x)))$ for all inputs. It is well known that the MIS problem is in NC [1, 3, 2, 4, 5, 20], and the LFMIS problem is one of the fundamental P-complete problems [9, 10, 11].

Recall that the EREW PRAM is the parallel model where the processors operate synchronously and share a common memory, but no two of them are allowed simultaneous access to a memory cell (whether the access is for reading or for writing in that cell). The CRCW PRAM differs from the EREW PRAM in that both simultaneous reading and simultaneous writing to the same cell are allowed; in case of simultaneous writing, the processor with lowest index succeeds. It is well known that a problem is in NC^{k+1} if it is solved in $O(\log^k n)$ time using a polynomial number of processors on a CRCW or EREW PRAM (see [21, Section 5.3]).

3 The measure LDPL

In [12], Uehara introduced a measure of the sequentiality for the lexicographically first maximal problems as follows: A *directed tree* T in G is the edge-induced subgraph of G such that T is a tree, and every path from the root to each leaf is a directed path on $\vec{T}$. The *maximum directed tree size* (MDTS) of G, denoted by MDTS(G), is defined by the maximum number of vertices of a directed tree in G.

In this paper, we introduce another measure. For $v_1, v_2, \cdots, v_k \in V$, $\alpha = (v_1, v_2, \cdots, v_k)$ is a *directed path* in $\vec{G} = (V, \vec{E})$ if $(v_i, v_{i+1}) \in \vec{E}$ for each $1 \leq i < k$. The *length* of α, denoted by $|\alpha|$, is defined by the number of edges in α. (In this case, $|\alpha| = k - 1$.) The *longest directed path length* (LDPL) of G, denoted by LDPL(G), is defined by the maximum length of directed paths in $\vec{G}$.

We first show that the LDPL is a better measure than the MDTS. It is not difficult to see that the MDTS equals to $\max\{|R(v)|\}$, where $R(v)$ is the set of vertices that are reachable from v on $\vec{G}$. Thus, the graph of LDPL l has the MDTS at least l. On the other hand, while a complete binary tree of size n has the MDTS n, it has the LDPL $\lceil \log n \rceil$. Hence, the graph family of LDPL $O(\log n)$ *properly* contains the graph family of MDTS $O(\log n)$. This implies that the LDPL is a better measure than the MDTS.

As noticed in [12], a "measure" is valid only if measuring the complexity of a problem is easier than solving the problem. We next state the following lemma to show it.

Lemma 1 [22] *The problem to compute* LDPL(G) *is in* NC^2.

Proof. We consider two languages L_1^k and L_2^k defined as follows; $L_1^k = \{[G,k] \mid G$ has a path of length at least $k\}$, and $L_2^k = \{[G,k] \mid$ all the paths in G are of length at most $k\}$. We first show that L_1^k is in NL. We assume that the input $[G,k]$ is encoded as

$$\#\mathtt{k}\#\#\#\mathtt{v}_0\#\mathtt{v}_{0,0}\#\mathtt{v}_{0,1}\#\cdots\#\mathtt{v}_{0,\mathtt{i}_0}\#\#\mathtt{v}_1\#\mathtt{v}_{1,0}\#\mathtt{v}_{1,1}\#\cdots\#\mathtt{v}_{1,\mathtt{i}_1}\#\#\mathtt{v}_2\cdots,$$

where $\mathtt{v}_{0,0}\#\mathtt{v}_{0,1}\#\cdots\#\mathtt{v}_{0,\mathtt{i}_0}$ is the list of vertices incident to the vertex v_0, and $v_{1,0}$, $v_{1,1}$, $\cdots$, v_{1,i_1} are vertices incident to the vertex v_1, and so on. An NTM M guesses the directed path of length k vertex by vertex. It nondeterministically starts from the vertex v_j, written as $\#\#\mathtt{v}_\mathtt{j}\#$ on the input tape, chooses one vertex greater than v_j from the list $\mathtt{v}_{\mathtt{j},0}\#\mathtt{v}_{\mathtt{j},1}\#\cdots\#\mathtt{v}_{\mathtt{j},\mathtt{i}_\mathtt{j}}$, and repeats the process until it can not find the next vertex. The machine M does not write the next vertex $v_{i',j'}$ itself on the work tape, for it can be too large. Instead, it remembers the position of $\#\mathtt{v}_{\mathtt{i}',\mathtt{j}'}\#$ on the input tape and looks for $\#\#\mathtt{v}_{\mathtt{i}',\mathtt{j}'}\#$ through all place of the form $\#\#\mathtt{x}\#$. To check whether $x = v_{i',j'}$, M compares x and $v_{i',j'}$ digit by digit. Through the computation, M also counts the number of the vertices, and accepts if it becomes k. Since $\vec{G}$ is acyclic, M visits each vertex on the input tape at most once. Thus M uses $O(\log n)$ space, where n is the length of the input. This establishes that $L_1^k \in NL$. It is not difficult to see that L_2^k is the

complement of L_1^{k+1}. Since NL is closed under complement [21, Theorem 4], L_2^k is also in NL. Thus, the language $L^k = \{[G,k] \mid \mathrm{LDPL}(G) = k\} = L_1^k \cap L_2^k$ is in NL and also in NC^2 since $NL \subseteq NC^2$ (see [21, Figure 7]). For each i with $1 \leq i \leq n$, we can make n parallel queries to compute whether $[G,i]$ is in L^i. This establishes the lemma. ∎

We here show the basic property for the LDPL:

Lemma 2 *Let $\alpha = (v_1, v_2, \cdots, v_k)$ be a directed path in $\vec{G}$ with $|\alpha| = \mathrm{LDPL}(G)$. Then $\mathrm{d}^-_{\vec{G}}(v_1) = 0$.*

Proof. To derive contradictions, assume that $\mathrm{d}^-_{\vec{G}}(v_1) > 0$. Then there is a vertex u with (u, v_1) in $\vec{G}$. First we assume that u is in α. Then the directed path from v_1 to u with the arc (u, v_1) makes a cycle, which contradicts the acyclicness of $\vec{G}$. Thus, u is not in α. Then, adding u to α, we get a directed path longer than α. This contradicts that $|\alpha| = \mathrm{LDPL}(G)$. ∎

Using Lemma 2, we have the following theorem:

Theorem 3 *The LFMIS problem on a graph of LDPL $O(\log^k n)$ is in NC^{k+1}, and the problem on a graph of LDPL $\Theta(n^\epsilon)$ is P-complete.*

Proof. In [12], Uehara showed that the LFMIS problem on a graph of MDTS $O(\log^k n)$ is in NC^{k+1}, and the problem on a graph of MDTS $\Theta(n^\epsilon)$ is P-complete. In the proof, he only use the fact that the root of the tree which gives the MDTS has indegree zero. Thus, using Lemma 2 instead of the fact, we have the theorem. ∎

4 Lexicographically first maximal subgraph problem for a local property

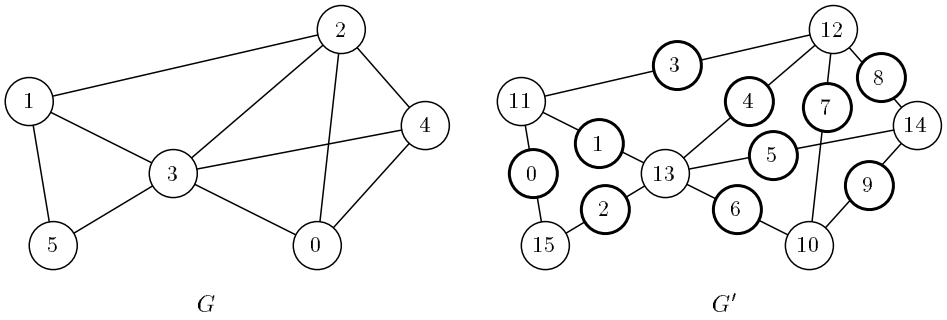
Miyano proved the P-completeness of the lexicographically first maximal subgraph (LFMS) problem for a hereditary property [10], and Shoudai and Miyano showed that the maximal subgraph problem for a local property is in NC [8]. Using their techniques, Uehara claimed that we can generalize the results for the LFMIS problem [12]. However, his proof is incomplete. We show a counterexample in this section.

For a given graph $G = (V, E)$, a subset U of V is called a *star* if $G[U]$ is acyclic and its diameter is at most 2. The vertex v in a star set U is a *center* of U if $N_{G[U]}(v) \cup \{v\} = U$. A subset U of V is called a *star set* if $G[U]$ only contains disjoint stars. A *maximal star set* (MSS) in G is a star set that is not properly contained in any other star set. The *lexicographically first maximal star set* (LFMSS) in G is the MSS I in G such that $I < J$ for every MSS J in G. The LFMSS problem is to find, given a graph G, the LFMSS in G. A star set satisfies the following properties stated in [10, 8, 12]; infinitely many graphs are star set, and some graph is not a star set (nontrivial); vertex induced subgraph of a star set is also a star set (hereditary); and independent edge set is a star set. Moreover, for a given graph $G = (V, E)$ and a subset U of V, testing if U is a star set in G can be solved in constant time using $n + m$ processors (assigned to each vertex and edge) on a CRCW PRAM by the following algorithm (its implement is not difficult and omitted here):

1. In parallel, each edge of which both endpoints are in U marks both endpoints;
2. In parallel, each vertex marked by two or more edges in step 1 is labeled with "center";
3. If there exists at least one edge of which both endpoints are "center" then the answer is "No", else the answer is "Yes."

We remark that an independent edge is a star and its endpoints are both centers. However, the edge could not be checked in step 3 since both endpoints were not labeled with "center" in step 2.

Theorem 4 *The LFMSS problem is P-complete even on a graph of LDPL 1 and MDTS 3.*



G G'

Figure 1: Reduction from LFMIS to LFMSS

Proof. We reduce the LFMIS problem to the LFMSS problem. For a given graph $G = (V, E)$ with n vertices and m edges, we construct a new graph $G' = (V', E')$ with $n + m$ vertices and $2m$ edges. We first replace each edge in E by a path of length 2. The replacement adds the graph m new vertices. The original vertices i in V are renumbered by $i + m$, and the new vertices are numbered by 0 to $m - 1$ in arbitrary order. Figure 1 depicts an example. Clearly, this is a log space reduction.

We first show that $\mathrm{LDPL}(G') = 1$ and $\mathrm{MDTS}(G') = 3$. In G', every path of length 2 forms either (new vertex, original vertex, new vertex) or (original vertex, new vertex, original vertex). On the other hand, each original vertex is greater than each new vertex. This implies that $\vec{G'}$ contains no directed paths of length 2. Thus, $\mathrm{LDPL}(G') = 1$. Moreover, every original vertex has no greater neighbors, and every new vertex has two greater neighbors. This immediately implies that $\mathrm{MDTS}(G') = 3$.

We next show that the algorithm for LFMSS on G' also finds the LFMIS on G. We consider the LFMSS U on G': It is not difficult to see that U must contain independent set $\{0, 1, \cdots, m-1\}$. That is, U must contain all new vertices. We here show that U contains no pair of original vertices that are joined in G. To derive a contradiction, we suppose that for two vertices u and v with $\{u, v\} \in E$ and U contains both of them. (We sometimes ignore that the index of u in G is changed to $u + m$ in G' for ease to read.) Here, without loss of generality, we assume that G has at least 3 vertices. Since G is connected, u or v has another neighbor. We say that w is the other neighbor of u. Then, $G'[U]$ is not a star set of G' since four vertices (the new vertex on the edge $\{w, u\}$, the original vertex u, the new vertex on the edge $\{u, v\}$, and the original vertex v) produce a path of length 3. Thus if U is a star set then $U - \{0, 1, \cdots, m-1\}$ corresponds to an independent set of G. It is easy to see that the maximality of the star set U on G' directly corresponds to the maximality of the independent set $U - \{0, 1, \cdots, m-1\}$ on G. From the rule of the reduction, it is also easy to see that U is the LFMSS of G' if and only if $U - \{0, 1, \cdots, m-1\}$ is the LFMIS on G. ∎

Remark: In [12], for a given graph G, Uehara claimed that his greedy parallel algorithm solves the LFMS problem in the time bounded above by $\mathrm{MDTS}(G)$. However, he missed the case that the auxiliary edges produce the long directed path. We briefly view how his algorithm misses to efficiently find the LFMSS of G' constructed in the proof above, although $\mathrm{MDTS}(G') = 3$. The algorithm, for given G', first finds the LFMIS $\{0, 1, \cdots, m-1\}$ in a constant time, and then, it adds auxiliary edges that produce the original graph G in this case. In the time, the algorithm faces the problem on the graph G of large MDTS, and can not solve it efficiently.

5 LDPL on a random graph

In this section, we consider the average value of the LDPL on a random graph. We denote by $Q(l)$ the graph property that the LDPL of the graph is at least l. We assume that $l < n$ without loss of generality. We only consider the case that l is an increasing function of n. The property $Q(l)$ seems to be similar to

the graph property that contains a path of length l, say $Q'(l)$. However, interestingly, these properties are completely different, and so their threshold values are. While the threshold value for $Q'(l)$ is $n^{-(l+1)/l}$ [15, Figure 13.1], the threshold value for $Q(l)$ is $\frac{l}{n}$ stated below. The reason is, intuitively, stated as follows: Let K_n be the complete graph with n vertices. Then it is not difficult to see that the number of the *undirected* (vertex-disjoint) paths of length l in K_n is $\binom{n}{l+1}\frac{(l+1)!}{2}$. On the other hand, the number of the *directed* paths on $\vec{K}_n$ is only $\binom{n}{l+1}$, since a directed path using specific $(l+1)$ vertices is uniquely determined following their order. This observation implies the following proposition for a random graph $G(n,p)$.

Proposition 5 *Let $N(n,p,l)$ be the expected value of the number of the directed paths of length l on $\vec{G}(n,p)$ with $l \geq 1$. Then $N(n,p,l) = \binom{n}{l+1}p^l$.*

Let l_0 be the least integer that satisfies $N(n,p,l_0) \leq 1$. Intuitively speaking, l_0 gives the LDPL on $G(n,p)$ on the average case. Using the equation $\left(\frac{n}{l}\right)^l \leq \binom{n}{l}$ (see [23, Proposition B.2] for example), we get $1 \geq \binom{n}{l_0+1}p^{l_0} \geq \left(\frac{n}{l_0+1}p\right)^{l_0}$, consequently, $np - 1 \leq l_0$. On the other hand, using the equation $\binom{n}{l} \leq \left(\frac{en}{l}\right)^l$ and $N(n,p,l_0-1) > 1$, we also get $1 < \binom{n}{l_0}p^{l_0-1} \leq \left(\frac{en}{l_0}\right)^{l_0} p^{l_0-1}$. This implies that $p < \left(\frac{enp}{l_0}\right)^{l_0}$, consequently, $l_0 < \frac{enp}{p^{\frac{1}{l_0}}} = enp^{1-\frac{1}{l_0}} < enp + 1$ for sufficient large n (or l_0). Thus the LDPL on $G(n,p)$ is nearly equal to np on the average case. We show that the average case occurs with high probability.

Lemma 6 *Let d be any positive integer with $d + l_0 \leq n$. Then $N(n,p,d+l_0) \leq (1-p)^d \left(\frac{l_0+1}{l_0+2}\right)^d$.*

Proof. Using $\binom{n}{l_0+1}p^{l_0} \leq 1$ and $l_0 \geq np - 1$, we have

$$\begin{aligned}
N(n,p,l_0+d) &= \binom{n}{l_0+d+1}p^{l_0+d} \\
&= \frac{n-l_0-1}{l_0+2}\frac{n-l_0-2}{l_0+3}\cdots\frac{n-l_0-d}{l_0+d+1}\binom{n}{l_0+1}p^{l_0}p^d \\
&\leq \frac{n-l_0-1}{l_0+2}\frac{n-l_0-2}{l_0+3}\cdots\frac{n-l_0-d}{l_0+d+1}\left(\frac{l_0+1}{n}\right)^d \\
&= \frac{n-l_0-1}{n}\frac{n-l_0-2}{n}\cdots\frac{n-l_0-d}{n}\frac{l_0+1}{l_0+2}\frac{l_0+1}{l_0+3}\cdots\frac{l_0+1}{l_0+d+1} \\
&\leq \left(\frac{n-l_0-1}{n}\right)^d\left(\frac{l_0+1}{l_0+2}\right)^d \leq (1-p)^d\left(\frac{l_0+1}{l_0+2}\right)^d.
\end{aligned}$$

∎

Lemma 7 *We assume that $\frac{n}{p} = o(n^2)$. Let c_1 be any fixed real number with $0 < c_1 < 1$, and l_1 be c_1np. Let ϵ be any given real number with $0 < \epsilon < 1$. Then for sufficiently large n, the LDPL of $G(n,p)$ is at least l_1 with probability greater than $1-\epsilon$.*

Proof. We consider the following algorithm:

1: Set $v := 0$;

2: Find the least vertex u such that $u > v$ and (v,u) is an arc in $\vec{G}(n,p)$;

3: If such u does not exist or the length of the directed path produced by the computed vertices is l_1, halt;

4: Set $v := u$, and go to step 2.

We show that the algorithm finds a directed path of length l_1 with high probability. This implies that the LDPL of $G(n,p)$ is at least l_1. Let X be a random variable defined to be the number of trials required to check whether (v,u) is an arc in $\vec{G}(n,p)$ in step 2. Let $C_1, C_2, \cdots, C_X$ denote the sequence of trials, where C_i is "*success*" if (v,u) is an arc in $\vec{G}(n,p)$, or "*failure*" otherwise. We assume that the algorithm halts because it finds a directed path of length l_1. In this case, C_X is success. We divide the sequence into *epochs* as follows; epoch 1 begins with C_1 and ends with the first success; and epoch i with $i > 1$ begins with the trial following the ith success and ends with the trial on which we obtain the $(i+1)$st success. Define the random variable X_i, for $1 \leq i \leq l_1$, to be the number of trials in the ith epoch, so that $X = \sum_{i=1}^{l_1} X_i$. The random variable X_i is geometrically distributed with parameter p since the probability of success on any trial is p from the definition of a random graph $G(n,p)$. Thus, the expected value of X_i is $\frac{1}{p}$, and its variance is $\frac{1-p}{p^2}$. Hence, the expected value of X is $\frac{l_1}{p} = c_1 n$, and its variance is $\frac{1-p}{p} c_1 n$. Thus, using Chebyshev's Inequality, we get

$$\Pr\left[\, |\, X - c_1 n\, |\, \geq \sqrt{\frac{(1-p)c_1 n}{(1-\epsilon)p}} \right] \leq 1 - \epsilon.$$

Here $\sqrt{\frac{(1-p)c_1 n}{(1-\epsilon)p}} < \sqrt{\frac{c_1}{1-\epsilon}}\sqrt{\frac{n}{p}} = o(n)$. Hence $c_1 n + \sqrt{\frac{(1-p)c_1 n}{(1-\epsilon)p}} < n$ for sufficiently large n. Consequently, the algorithm success to find a path of length l_1 with probability greater than $1-\epsilon$ for sufficiently large n. This completes the proof. ∎

Lemma 8 *For a given random graph $G(n,p)$ and a positive integer l, suppose $N(n,p,l) < 1$. Then $Q(l)$ is satisfied with probability at most $N(n,p,l)$.*

Proof. While $N(n,p,l)$ is $\sum_{i=1}^{\infty} i \Pr[G \text{ contains } i \text{ directed paths of length } l]$,
$\Pr[Q(l) \text{ is satisfied}]$ is equal to $\sum_{i=1}^{\infty} \Pr[G \text{ contains } i \text{ directed paths of length } l]$.
This immediately implies the lemma. ∎

Now we are ready to show the threshold function of $Q(l)$. We remind that the function $r(n)$ is the threshold function of $Q(l)$ if $\lim_{n\to\infty} p(n)/r(n) = 0$ implies that almost no graph has property $Q(l)$, and $\lim_{n\to\infty} p(n)/r(n) = \infty$ implies that almost every graph has property $Q(l)$.

Theorem 9 *For $Q(l)$ with $l \geq 1$, $r(n,l) = \frac{l}{n}$ is the threshold function.*

Proof. Let l_2 be the least integer that satisfies $N(n,r,l_2) \leq 1$ on a random graph $G(n,r)$. Then $enr + 1 \geq l_2 \geq nr - 1$.

We first consider a random graph $G(n,p)$ such that $\lim_{n\to\infty} p(n,l)/r(n,l) = \infty$. Then for any positive fixed constant c_3 with $0 < c_3 < 1$, $c_3 np > enr + 1 \geq l_2$ for sufficiently large n. This implies that $\frac{n}{p} < \frac{n^2}{l_2} = \frac{n^2}{c_2 l} = o(n^2)$. Thus, from Lemma 7, the LDPL of $G(n,p)$ is at least l_2 with probability greater than $1-\epsilon$ for any fixed positive real number ϵ.

Next assume that $\lim_{n\to\infty} p(n,l)/r(n,l) = 0$. Let l_3 be the least integer that satisfies $N(n,p,l_3) \leq 1$ on a random graph $G(n,p)$. Since $enp + 1 \geq l_3$, for any fixed positive integer d, $(1+d)l_3 \leq (1+d)(enp+1) < nr - 1 \leq l_2 < n$ for sufficiently large n. From Lemma 6, $N(n,p,l_3 + dl_3) \leq (1-p)^{dl_3} \left(\frac{l_3+1}{l_3+2}\right)^{dl_3}$. When p is a positive constant, $N(n,p,l_3 + dl_3)$ converges to 0 since $N(n,p,l_3 + dl_3) \leq (1-p)^{dl_3}$. When $p = o(1)$, consequently, $l_3 = o(n)$, we have $N(n,p,l_3 + dl_3) \leq \left(\frac{l_3+1}{l_3+2}\right)^{dl_3} = \left(1 - \frac{1}{l_3+2}\right)^{d(l_3+2)} \left(\frac{l_3+2}{l_3+1}\right)^{2d} \leq \left(\left(1 + \frac{1}{l_3+1}\right)^2 \frac{1}{e}\right)^d$, which also converges to 0 for large d and n (or l_3). This together with Lemma 8 implies that the probability that the LDPL of $G(n,p)$ is at least l_3 converges to 0. Since $l_3 < l_2$, the LDPL of $G(n,p)$ is at least l_2 with probability less than ϵ for any fixed positive real number ϵ. ∎

6 Concluding Remarks

Although the LDPL is better than the MDTS, the LDPL does not completely characterize the parallel complexity of the LFMIS problem. For any fixed positive real number $0 < p < 1$, while the LDPL of a random graph $G(n,p)$ is $\Theta(n)$ with high probability, the LFMIS problem on the random graph can be solved in $O(\log n)$ time (on average) in parallel [16, 17]. That is, there seems to exist a gap between the measure LDPL and the time required to solve the LFMIS on the random graph. Moreover, as stated in Section 4, we have the lexicographically first maximal subgraph problem whose parallel complexity is not characterized by the LDPL (and the MDTS). These facts motivate us to find a *better* measure that characterize the parallel complexity of P-complete problems.

Acknowledgements

I thank Professor Osamu Watanabe, who noticed an error for the parallel complexity of the LDPL problem in an early draft. I indebted to Professor Seinosuke Toda and Professor Zhi-Zhong Chen for their many helpful comments. Professor Seinosuke Toda is the first man who noticed that we can use the LDPL as another measure. Professor Zhi-Zhong Chen gave many fruitful suggestion for the complexity of computing the measure. My thanks also to Dr. Koichi Yamazaki for discussions and encouragement.

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