Completeness of Graph Isomorphism Problem for Bipartite Graph Classes

Takayuki Nagoya†,Ryuhei Uehara‡*, Seinosuke Toda§

† Department of Computer Science and Information,
The University of Electro-Communications
‡ Natural Science Faculty, Komazawa University
§ Department of Computer Science and System Analysis,
College of Humanities and Sciences, Nihon University

Abstract

It is well known that the graph isomorphism (GI) problem is GI complete even for the bipartite graphs. Recently, it is shown that the GI problem can be solved in polynomial time for the convex graphs, which is a subclass of bipartite graphs. Chordal bipartite graphs form a well-known graph class between bipartite graphs and convex graphs. The relative complexity of the GI problem for the class is unknown. We show that the GI problem is GI complete even for the chordal bipartite graphs, which draws a line between the convex graphs and chordal bipartite graphs.

Key words: computational complexity, chordal bipartite graph, graph isomorphism complete, graph isomorphism problem.

1 Introduction

The graph isomorphism (GI) problem is a well-known open problem and was listed as an important open problem in Karp [6] over two decades ago. Although the problem is trivially in NP, the problem is not known to be in P and not known to be NP-complete either (comprehensive references can be found in [9], and see [7] for further details).

A graph class is called *graph isomorphism (GI) complete* if the GI problem for the graph class is polynomial time equivalent to the GI problem for general graphs. There are several GI complete graph classes including regular graphs, bipartite graphs, line graphs, rooted acyclic digraphs, comparability graphs, split graphs, k-trees (for unbounded k), undirected path graphs (see [2] for a review on GI complete graph classes).

On the other hand, it has been shown that the GI problem is solvable in polynomial time when it is restricted to special graph classes. For example, graphs of bounded degrees, planar graphs, trees, interval graphs, permutation graphs, k-trees (for fixed k), transitive series parallel digraphs, graphs of bounded genus, and graphs of bounded eigenvalue multiplicity (see [1] for a comprehensive reference for the results).

Recently, many graph classes have been proposed and widely investigated. (There is a good survey for the graph classes by Brandstädt, Le, and Spinrad [3].) In [1], Babel, Ponomarenko and Tinhofer show that the GI problem for directed path graphs is GI complete, while the GI problem for rooted directed path graphs is polynomial time solvable. Their results draw a line between interval graphs and chordal graphs, because we have the following inclusion (for graph classes not explicitly defined in this paper we refer to [1, 3]):

$$\text{interval graphs} \subset \text{rooted directed path graphs} \subset \text{directed path graphs}$$

$$\subset \text{undirected path graphs} \subset \text{chordal graphs}.$$

*Work done while visiting University of Waterloo.

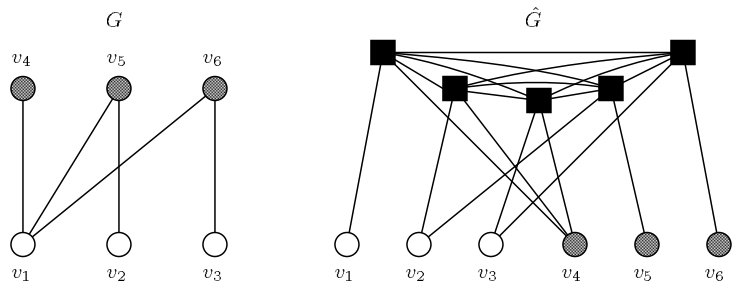

Figure 1: Reduction from bipartite graph to directed path graph

We now turn to the GI problem for some subclasses of bipartite graphs. We know that the GI problem for the bipartite graphs is GI complete, and recently, Chen shows that the GI problem for convex graphs is polynomial time solvable [4]. On the other hand, we have the following inclusion:

$$\text{convex graphs} \subset \text{chordal bipartite graphs} \subset \text{bipartite graphs}.$$

However, the relative complexity of the GI problem for chordal bipartite graphs is unknown. In this paper, we show that the line lies between convex graphs and chordal bipartite graphs: the GI problem is GI complete even for chordal bipartite graphs.

2 Preliminaries

A graph $G = (V, E)$ consists of a finite set V of *vertices* and a collection E of 2-element subsets of V called *edges*. The *neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V | \{u, v\} \in E\}$, and the *degree* of a vertex v is $|N_G(v)|$ and denoted by $d_G(v)$. If no confusion can arise we will omit the index G. For the vertex set U of V, we denote by $G[U]$ the subset of G induced by U. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic* if and only if there is a one-to-one mapping $\phi : V \to V'$ which satisfies $\{u, v\} \in E$ if and only if $\{\phi(u), \phi(v)\} \in E'$ for every pair of vertices u and v. We denote by $G \sim G'$ if G and G' are isomorphic. Given graphs G and G', *graph isomorphism (GI) problem* is the problem to determine if $G \sim G'$.

For a given graph $G = (V, E)$, a sequence of the vertices $v_0, v_1, \cdots, v_l$ is a *path*, denoted by $(v_0, v_1, \cdots, v_l)$, if $\{v_j, v_{j+1}\} \in E$ for each $0 \leq j \leq l - 1$. The *length* of a path is the number of edges on the path. For two vertices u and v, the *distance* of the vertices is the minimum length of the paths joining u and v. A *cycle* is a path beginning and ending with the same vertex. An edge which joins two vertices of a cycle but is not itself an edge of the cycle is a *chord* of that cycle. A graph $G = (V, E)$ is *bipartite* if V can be divided into two sets X and Y such that every edge joins a vertex in X and another vertex in Y. A graph is *chordal bipartite* if the graph is bipartite and every cycle of length at least 6 has a chord.

3 Reduction

The main theorem in this paper is the following.

Theorem 1 The GI problem for chordal bipartite graphs is GI complete.

Babel, Ponomarenko and Tinhofer show that the GI problem for directed path graphs is GI complete in [1]. They give a reduction from any given bipartite graph to a directed path graph such that two given bipartite graphs are isomorphic if and only if reduced directed path graphs are isomorphic. In order to be self-contained, we briefly show their reduction. Given bipartite graph $G = (X, Y, E)$ with $|X \cup Y| = n$ and $|E| = m$, they construct a graph $\hat{G} = (\hat{V}, \hat{E})$ as follows: $\hat{V} = X \cup Y \cup E$, and $\hat{E}$ contains (1) $\{e, e'\}$ for all e, e' in E, (2) $\{x, e\}$ for each $x \in X$ and $e \in E$ with $x \in e$, that is, $e = \{x, y\}$ for some $y \in Y$, and (3) $\{y, e\}$ for each $y \in Y$ and $e \in E$ with $y \in e$, that is, $e = \{x, y\}$ for some $x \in X$. An illustrating example is given in Fig. 1 (Each black square vertex of $\hat{G}$ corresponds to an edge of G.). They show that any given bipartite

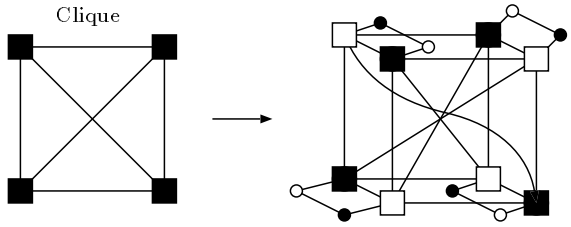

Figure 2: Reduction from a clique

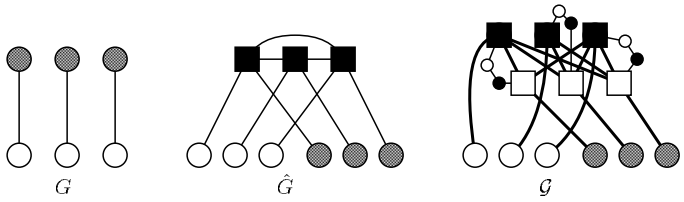

Figure 3: Reduction from bipartite graph G to chordal bipartite graph $\mathcal{G}$

graph G, $\hat{G}$ is a directed path graph. Our start point is the directed path graph $\hat{G}$. The directed path graph $\hat{G} = (X \cup Y \cup E, \hat{E})$ is a very restricted split graph [1] . By their reduction, $\hat{G}$ has the following properties; (a) $\hat{G}[E]$ is a clique of size m, (b) $\hat{G}[X \cup Y]$ is an independent set of size n, and (c) for each $e \in E$, e has exactly one neighbor in X, and another neighbor in Y. (Thus we also have each vertex $e \in E$ has degree $m + 1$.) Without loss of generality, by technical reason stated later, we also assume that (d) $m > 1$ and (e) $|X|, |Y| > 1$.

We now turn to our reduction from the directed path graph $\hat{G} = (X \cup Y \cup E, \hat{E})$ with the properties to a chordal bipartite graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. We let $\mathcal{V} = X \cup Y \cup E \cup E' \cup B \cup W$. Each vertex $e \in E$ corresponds to three vertices $e' \in E', e_b \in B$, and $e_w \in W$, respectively. That is, $|E| = |E'| = |B| = |W| = m$. We first show how to connect the vertices in $E \cup E' \cup B \cup W$. (1) for each vertex $e \in E$, four edges $\{e, e'\}$, $\{e', e_b\}$, $\{e_b, e_w\}$, and $\{e, e_w\}$ are added into $\mathcal{E}$, and (2) for each pair of vertices e_1 and e_2, we add $\{e_1, e_2'\}$ and $\{e_1', e_2\}$ into $\mathcal{E}$. We remind that $\hat{G}[E]$ is a clique. In other words, $\mathcal{G}[E \cup E']$ is a bipartite complete graph. An illustrating example is given in Fig. 2 (black square vertices are in E, white square vertices are in E', small black vertices are in B, and small white vertices are in W). We next show how to connect the vertices in X and Y to the vertices in $E \cup E' \cup B \cup W$. First mention that the vertices in $B \cup W$ are not connected any vertices in $X \cup Y$. The vertex set $B \cup W$ will be used as handles to reconstucut the original graph. We now complete the reduction; for each vertex $x \in X$, we add $\{x, e\}$ into $\mathcal{E}$ if $\{x, e\} \in \hat{E}$; and for each vertex $y \in Y$, we add $\{y, e'\}$ into $\mathcal{E}$ if $\{y, e\} \in \hat{E}$. An illustrating example is given in Fig. 3. Given bipartite graph G, the reduced graph $\mathcal{G}$ has $n + 4m$ vertices and $m^2 + 5m$ edges. It is easy to see that the reduction can be done in polynomial time. In order to complete the proof of the main theorem, we have to show two lemmas.

Lemma 2 Given bipartite graph $G = (X, Y, E)$, the reduced graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is chordal bipartite.

Proof. We divide the vertex set $\mathcal{V} = X \cup Y \cup E \cup E' \cup B \cup W$ into $\mathcal{V}_1 = X \cup E' \cup W$ and $\mathcal{V}_2 = Y \cup E \cup B$. It is easy to see that every edge in $\mathcal{E}$ joins one vertex in $\mathcal{V}_1$ and the other one in $\mathcal{V}_2$ (intuitively, each edge joins white vertex with black or gray vertex in Fig. 3). Thus $\mathcal{G}$ is bipartite. Next we show every cycle of length at least 6 has a chord. To derive a contradiction, we assume that we have a chordless cycle C of length at least 6. If C contains at least one vertex in $B \cup W$, we immediately have induced subgraph C_4. Thus C only contains vertices in $X \cup Y \cup E \cup E'$. Let v_0, v_1, v_2, v_3 are consecutive distinct vertices on C. If they are

[1] A graph $G = (V, E)$ is a split graph if V can be partitioned into two subsets V_1 and V_2 such that $G[V_1]$ is a clique and $G[V_2]$ is an independent set.

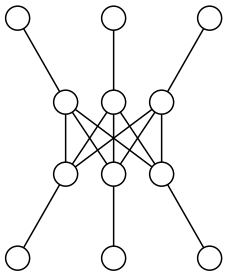

Figure 4: An insect

all in $E \cup E'$, since $\mathcal{G}[E \cup E']$ is bipartite complete, $\{v_0, v_3\} \in \mathcal{E}$. Thus at least one vertex is not in $E \cup E'$. Without loss of generality, we assume that v_1 is in X, and hence v_0 and v_2 are in E. Then, by the property (c), no other vertex in X is incident to v_0 and v_2. Since no vertices in Y are incident to E, v_3 is in E'. Hence we have $\{v_0, v_3\}$ in $\mathcal{E}$, which is a contradiction. Thus $\mathcal{G}$ is a chordal bipartite graph. ∎

We finally complete the proof of the main theorem:

Lemma 3 Given bipartite graph G_1 and G_2 are isomorphic if and only if $\mathcal{G}_1$ and $\mathcal{G}_2$ are isomorphic.

Proof. Given two bipartite graph G_1 and G_2, it is clear that $G_1 \sim G_2$ if and only if $\hat{G_1} \sim \hat{G_2}$. Thus we show that $\hat{G_1} \sim \hat{G_2}$ if and only if $\mathcal{G}_1 \sim \mathcal{G}_2$. Since the only if part is trivial, we consider the if part. Given graph G_1, we assume that $\mathcal{G}_1 \sim \mathcal{G}_2$ for some G_2. It is sufficient to show that we can reconstruct G_2, which is isomorphic to G_1, from $\mathcal{G}_2$ with no ambiguity.

By the properties (c) and (d), each vertex in $E \cup E'$ has degree at least 3. Each vertex in $X \cup Y$ has its neighbors only in $E \cup E'$. Thus, a vertex v is in $B \cup W$ if and only if $d(v) = 2$ and one neighbor u satisfies also $d(u) = 2$. Thus using the connectivity of $\mathcal{G}[B \cup W \cup E \cup E']$, we can reconstruct B' and W' such that either $B = B', W = W'$, or $B = W', W = B'$. Here, we first assume that $B = B'$ and $W = W'$. The only neighbors of W are in E', and only neighbors of B are in E. Thus we can reconstruct B, W, E, and E'. The other vertices are in $X \cup Y$, and each of them can be distinguished into X and Y; it is in X if it is adjacent to the vertices in E, and it is in Y if it is adjacent to the vertices in E'. To make $\hat{G_2}$, we have to join $e \in E$ to $e' \in E'$. To do this, we use the "handle" e_b in B and e_w in W. That is, for each pair $e_b \in B$ and $e_w \in W$ with $\{e_b, e_w\} \in \mathcal{E}$, there are two vertices $e \in E$ and $e' \in E'$ such that $\{e, e_w\}, \{e', e_b\} \in \mathcal{E}$. Hence we can get the vertex e in $\hat{G_2}$ by contracting the corresponding four vertices. After contracting every four-tuple of vertices in $B \cup W \cup E \cup E'$, we can reconstruct the graph $\hat{G_2}$, which is isomorphic to G_1.

We remind that we assume that $B = B'$ and $W = W'$. When $B = W'$ and $W = B'$, we obtain the different graph $\hat{G_2}'$. But it is easy to see that $\hat{G_2}'$ is isomorphic to $\hat{G_2}$; just swap X with Y. ∎

Now the main theorem follows immediately from Lemmas 2 and 3.

4 Concluding Remarks

As mentioned in Introduction, we have many graph classes, which are proposed recently, and we do not know whether the GI problem is GI complete or polynomial time solvable on some classes. In order to clarify the complexity of the GI problem, considering the GI problem on such graph classes is future work.

Among the subclasses of bipartite graphs, it is worth remarking that there is an interesting graph class between convex graphs and chordal bipartite graphs; interval bigraphs, which was introduced by Harary, Kabell, and McMorris [5]. Müller shows that

$$\text{convex graphs} \subset \text{interval bigraphs} \subset \text{chordal bipartite graphs},$$

and the class of interval bigraphs can be recognized in polynomial time [8].

Unfortunately, our reduction generates a graph which is not an interval bigraph: In [8, Lemma 9], Müller shows that an interval bigraph does not contain an induced subgraph isomorphic to an insect shown in Fig. 4. On the other hand, our reduction can generates a chordal bipartite graph which contain an induced subgraph

isomorphic to an insect (see the thick line in Figure 3). Hence our reduction does not show GI-completeness of interval bigraphs, and the complexity of the GI problem on interval bigraphs is open.

References

[1] L. Babel, I.N. Ponomarenko, and G. Tinhofer. The Isomorphism Problem For Directed Path Graphs and For Rooted Directed Path Graphs. *Journal of Algorithms*, 21:542–564, 1996.

[2] K.S. Booth and C.J. Colbourn. Problems Polynomially Equivalent to Graph Isomorphism. Technical Report CS-77-04, Computer Science Department, University of Waterloo, 1979.

[3] A. Brandstädt, V.B. Le, and J.P. Spinrad. *Graph Classes: A Survey.* SIAM, 1999.

[4] L. Chen. Graph Isomorphism and Identification Matrices: Sequential Algorithms. *Journal of Computer and System Sciences*, 59:450–475, 1999.

[5] F. Harary, J.A. Kabell, and F.R. McMorris. Bipartite intersection graphs. *Comment. Math. Univ. Carolin.*, 23:739–745, 1982.

[6] R.M. Karp. *Complexity of Computer Computations*, chapter Reducibility among Combinatorial Problems. Plenum, 1972.

[7] J. Köbler, U. Schöning, and J. Torán. *The Graph Isomorphism Problem: Its Structural Complexity.* Birkhäuser, 1993.

[8] H. Müller. Recognizing Interval Digraphs and Interval Bigraphs in Polynomial Time. *Disc. Appl. Math.*, 78:189–205, 1997.

[9] R.C. Read and D.G. Corneil. The Graph Isomorphism Disease. *Journal of Graph Theory*, 1:339–363, 1977.