Chapter 4

Complexity Classes Characterized by Semi-Random Sources

4.1 Probability of a worst tree

First, we show two crucial lemmas for a clustered tree.

Lemma 4.1 Let T be a clustered tree of depth d. Let $t_0, t_1, \cdots, t_k$ be the integers such that $|T| = \sum_{i=0}^{k} 2^{t_i}$ with $t_k > \cdots > t_1 > t_0 \geq 0$. Then, $\Pr[T|F_{\max}(T)] = \sum_{i=0}^{k} \delta^i (1-\delta)^{d-t_{k-i}-i}$.

Proof. From a δ-tree of depth d with no accepting leaf, we construct a δ-tree of depth d with $|T|$ accepting leaves using the following steps:

Step k: Let T_k be the rightmost subtree of depth t_k. Change all the labels of the leaves of T_k from "reject" to "accept".

Step i $(i = k-1, k-2, \cdots, 0)$: Let T'_{i+1} be the subtree such that its parent is also the parent of T_{i+1} defined in step $i+1$. Let T_i be the rightmost subtree of depth t_i of T'_{i+1}. (Note that this step works since $t_{i+1} > t_i$.) Change all the labels of the leaves of T_i from "reject" to "accept".

Since each T_i is always taken from rightmost side, we obtain a clustered tree with $|T|$ accepting leaves after the construction. Thus the constructed tree is the same tree as T. The path from the root of T to the root of T_i consists of the path from the root of T to the parent of T_k (whose $(d - t_k - 1)$ arcs are assigned $(1-\delta)$), an arc to the root of T'_k (which is assigned δ), the path to the parent of T_{k-1} (whose $(t_k - t_{k-1} - 1)$ arcs are assigned $(1-\delta)$), an arc to the root of T'_{k-1}, $\cdots$, and the path to the root of T_i. Thus, the probability of the root of T_i $(k \geq i \geq 0)$ is $\delta^i (1-\delta)^{d-t_{k-i}-i}$. Adding them together, we have the lemma. ∎

Lemma 4.2 Let l, r and d be any integers with $0 \leq l \leq r \leq 2^{d-1}$, and T be a clustered tree of depth d with $|T| = l + r$. Let T' be a δ-tree of depth d such that $|T'_l| = l$, $|T'_r| = r$, and both T'_l and T'_r are clustered trees. Then, $\Pr[T'|F_{\max}(T')] \geq \Pr[T|F_{\max}(T)]$.

Proof. We show the lemma by induction on the depth of the tree. Since it is clear when $d \leq 2$, we assume $d > 2$. Here, $\Pr[T'|F_{\max}(T')] = \delta^2 \Pr[T'_{ll}|F_{\max}(T')] + \delta(1-\delta)\Pr[T'_{lr}|F_{\max}(T')] + (1-\delta)\delta\Pr[T'_{rl}|F_{\max}(T')] + (1-\delta)^2 \Pr[T'_{rr}|F_{\max}(T')]$. Thus, the probability of T' does not change even if we exchange T'_{lr} and T'_{rl}. For T'_{ll} and T'_{rl}, four cases arise:
Case 1: $|T'_{ll}| > 0$ and $|T'_{rl}| = 0$. This case is impossible since $0 \leq l \leq r$.

Case 2: $|T'_{ll}| = |T'_{rl}| = 0$. Exchanging T'_{lr} and T'_{rl}, we find that only T'_r has accepting leaves with $|T'_{rl}| = l$ and $|T'_{rr}| = r$. Thus, by the inductive hypothesis, $\Pr[T'_r|F_{\max}(T'_r)] \geq \Pr[T_r|F_{\max}(T_r)]$. Since $|T_l| = |T'_l| = 0$, the lemma holds.
Case 3: $|T'_{ll}| > 0$ and $|T'_{rl}| > 0$. Since T'_l and T'_r are clustered trees, $|T'_{lr}| = |T'_{rr}| = 2^{d-2}$. Thus, exchanging T'_{lr} and T'_{rl}, we find that $|T'_r| = 2^{d-1}$. Moreover, $\Pr[T'_l|F_{\max}(T'_l)] \geq \Pr[T_l|F_{\max}(T_l)]$ by the inductive hypothesis. Since $|T_r| = |T'_r| = 2^{d-1}$, the lemma holds.
Case 4: $|T'_{ll}| = 0$ and $|T'_{rl}| > 0$. Here, T'_{llr}, T'_{lrl} and T'_{rll} are exchangeable each other, and so are T'_{lrr} and T'_{rlr}. For T'_{lrl} and T'_{rll}, four subcases arise:
Subcase 4a: $|T'_{lrl}| = |T'_{rll}| = 0$. The probabilities of the roots of T'_{lrr} and T'_{rll} are $\delta(1-\delta)^2$ and $\delta^2(1-\delta)$, respectively. Thus, since $|T'_{rll}| = 0$, even if we exchange T'_{lrr} and T'_{rll}, the probability of T' does not increase. Thus we have the lemma using the inductive hypothesis for T'_{rl}.
Subcase 4b: $|T'_{lrl}| > 0$ and $|T'_{rll}| = 0$. First, exchange T'_{lrr} and T'_{rlr}. Next, replace T'_{lr} by a clustered tree with $|T'_{lr}|$ accepting leaves. After replacing, if $|T'_{lrl}| = 0$, this case is reduced to Subcase 4a; otherwise, $|T'_{lrl}| > 0$, $|T'_{lrr}| = 2^{d-3}$, and $|T'_{rll}| = 0$. In the latter case, exchanging T'_{lrr} and T'_{rll} and further exchanging T'_{lrl} and T'_{lrr}, we see that T' is reduced to a clustered tree. Thus, it is sufficient to show that these exchanges do not increase the probability. Let $p = \Pr[T'_{lrl}|F_{\max}(T')]$ before the exchanges. Then the change of the probability by the exchanges is $-\delta(1-\delta)^2 + \delta^2(1-\delta) - \delta^2(1-\delta)p + \delta(1-\delta)^2 p = \delta(1-\delta)(1-2\delta)(p-1) < 0$, since $0 < \delta < \frac{1}{2}$ and $0 < p < 1$. This implies the lemma.
Subcase 4c: $|T'_{lrl}| = 0$ and $|T'_{rll}| > 0$. Since T'_{lr} and T'_{rl} are exchangeable, this case is reduced to Subcase 4b.
Subcase 4d: $|T'_{lrl}| > 0$ and $|T'_{rll}| > 0$. First, exchange T'_{llr} and T'_{rll}. Since $|T'_{ll}| = 0$ before the exchange, $|T'_{rll}| = 0$ after the exchange. Next, replace T'_l by a clustered tree with $|T'_l|$ accepting leaves. After replacing, if $|T'_{llr}| = 0$, this case is reduced to Subcase 4b; otherwise, $|T'_{llr}| > 0$ and $|T'_{lr}| = 2^{d-2}$. In the latter case, exchanging T'_{lrl} and T'_{rll} and further exchanging T'_{llr} and T'_{lrl}, we get a clustered tree without changing the probability. ∎

Now, we are ready to show the main theorem in this section.

Theorem 4.3 Every clustered tree is worst.

Proof. We show the theorem by induction on the depth of the tree. Since it is clear when $d \leq 1$, we assume that $d > 1$. Let T be a given clustered tree of depth d, and T' be any worst tree of depth d with $|T'| = |T|$. Since T' is worst, it is sufficient to show that $\Pr[T'|F_{\max}(T')] \geq \Pr[T|F_{\max}(T)]$. If either T'_l or T'_r is not worst, we can improve the probability of T' by replacing them. This contradicts that T' is worst. Thus, both T'_l and T'_r are worst. By the inductive hypothesis, we can replace T'_l and T'_r by clustered trees with $|T'_l|$ and $|T'_r|$ accepting leaves, respectively. Thus, by Lemma 4.2, $\Pr[T'|F_{\max}(T')] \geq \Pr[T|F_{\max}(T)]$. ∎

Corollary 4.4 Let T be a worst tree of depth d. Then, $\Pr[T|F_{\max}(T)] = \sum_{i=0}^{k} \delta^i (1-\delta)^{d-t_{k-i}-i}$, where $|T| = \sum_{i=0}^{k} 2^{t_i}$ with $t_k > \cdots > t_1 > t_0 \geq 0$.

4.2 The class δ-PP

Theorem 4.5 δ-PP = BPP for $0 < \delta < \frac{1}{2}$.

Proof. By Theorem 2.20, Definitions 2.13 and 2.14, BPP = δ-BPP $\subseteq$ δ-PP. Thus it is sufficient to show that δ-PP $\subseteq$ BPP. Let M_1 be a PTM with $L(M_1) \in \delta$-PP for some δ, and $p(n)$ be the depth of the computation path of M_1 on input of length n. Let m be a minimum positive integer such that $(1-\delta)(1-\delta^m) \geq \frac{1}{2}$. Since $\lim_{m\to\infty}(1-\delta)(1-\delta^m) = 1-\delta > \frac{1}{2}$, m must be a constant.

Without loss of generality, we assume that $p(n) > m+1$. Here we consider a worst tree T of depth $p(n)$ with $|T| \geq 2^{p(n)-1} - 2^{p(n)-m-1}$. Since $2^{p(n)-1} - 2^{p(n)-m-1} = \sum_{i=1}^{m} 2^{p(n)-i-1}$, using Corollary 4.4, we find that $\Pr[T|F_{\max}(T)] = (1-\delta)^2(\delta^0+\delta^1+\cdots+\delta^{m-1}) = (1-\delta)(1-\delta^m) \geq \frac{1}{2}$.

First, we consider an x_1 of length n with $x_1 \notin L(M_1)$. Since $x_1 \notin L(M_1)$, for its computation tree T_1 and every δ-assignment F to T_1, $\Pr[T_1|F] < \frac{1}{2}$. Consequently, $\Pr[T_1|F_{\max}(T_1)] < \frac{1}{2}$. Thus, Proposition 2.26 shows that $|T_1| < \frac{1}{2} \leq |T| = 2^{p(n)-1} - 2^{p(n)-m-1}$. Hence if M_1 computes with a perfect random source, $\Pr[M_1 \text{ accepts } x_1] < \frac{2^{p(n)-1}-2^{p(n)-m-1}}{2^{p(n)}} = \frac{1}{2} - \frac{1}{2^{m+1}}$. Since m is a constant, by repeating the algorithm enough times, the probability can be improved to $\frac{1}{4}$.

Second, we consider an x_2 of length n with $x_2 \in L(M_1)$. For the computation tree T_2 of x_2, we denote by $\bar{T}_2$ the computation tree constructed by replacing each accepting (rejecting) leaf by a rejecting (resp. accepting) leaf. Then, $\Pr[T_2|F] > \frac{1}{2}$ for every δ-assignment F to T_2, consequently, $\Pr[\bar{T}_2|F_{\max}(\bar{T}_2)] < \frac{1}{2}$. Thus, using the same argument above, we have $|\bar{T}_2| < 2^{p(n)-1} - 2^{p(n)-m-1}$, or $|T_2| > 2^{p(n)} - |\bar{T}_2| = 2^{p(n)-1} + 2^{p(n)-m-1}$. Hence if M_1 computes with a perfect random source, $\Pr[M_1 \text{ accepts } x_2] > \frac{1}{2} + \frac{1}{2^{m+1}}$. This completes the proof. ∎

4.3 The class $^{\exists}\delta$-RP

Theorem 4.6 $^{\exists}\delta\text{-RP} = \mathsf{NP}$ for $0 < \delta < \frac{1}{2}$.

Proof. From Proposition 2.18, it is sufficient to show that $\mathsf{NP} \subseteq {}^{\exists}\delta\text{-RP}$. Let M_2 be an NTM with $L(M_2) \in \mathsf{NP}$, and $p(n)$ be the length of M_2's computation on input of length n. Let $q(n) = \lceil -\frac{\log(8(p(n)+1))}{\log(2\sqrt{\delta(1-\delta)})} \rceil$. Since $\log(2\sqrt{\delta(1-\delta)}) < 0$ for $0 < \delta < \frac{1}{2}$, $q(n) > 0$. We construct a PTM M_2' which simulates M_2, with a specific δ-random source. M_2' simulates M_2 straightforwardly if M_2 is not in a nondeterministic state. Otherwise, M_2' simulates M_2 as follows: When M_2 nondeterministically selects 0-choice (resp. 1-choice), M_2' assigns $(1-\delta)$ to the probability that the outcome of a coin tossing is 0 (resp. 1); tosses a coin $q(n)$ times; and selects i-choice, where i is the majority of the outcomes.

It is clear that M_2' simulates M_2 in time polynomial in n, and M_2' rejects x for $x \notin L$. We consider the probability that M_2' accepts x for $x \in L$. M_2' fails to simulate one nondeterministic choice of M_2 with probability equal to $\sum_{i=0}^{\frac{1}{2}q(n)} \binom{q(n)}{i} \delta^{q(n)-i}(1-\delta)^i$. Since $0 < \frac{1}{2} < (1-\delta) < 1$, we have $\sum_{i=0}^{\frac{1}{2}q(n)} \binom{q(n)}{i} \delta^{q(n)-i}(1-\delta)^i \leq 2^{q(n)\log(2\sqrt{\delta(1-\delta)})} \leq 2^{-\log(8(p(n)+1))} = \frac{1}{8(p(n)+1)}$ using the lemma due to Luts [Lut90] that can be proved using Chernoff bounds [Che52]. Thus the probability that M_2' successes in simulating at most $p(n)$ nondeterministic choices of M_2 is greater than or equal to $\left(1 - \sum_{i=0}^{\frac{1}{2}q(n)} \binom{q(n)}{i} \delta^{q(n)-i}(1-\delta)^i\right)^{p(n)} \geq \left(1 - \frac{1}{8(p(n)+1)}\right)^{p(n)}$. Here, $e^{-p} < \left(1 - \frac{p}{n+1}\right)^n$ holds for $0 < p < 1$ and any positive integer n. Thus, M_2' accepts x for $x \in L$ with probability greater than or equal to $\left(1 - \frac{1}{8(p(n)+1)}\right)^{p(n)} > e^{-\frac{1}{8}} > \frac{1}{2}$. Consequently, $L \in {}^{\exists}\delta\text{-RP}$. ∎

Intuitively, Theorem 4.6 shows that the PTM using polynomial number of semi-random bits generated by a specific δ-random source can simulate polynomial number of nondeterministic choices with high probability.

4.4 The classes $^{\exists}\delta$-BPP and $^{\exists}\delta$-PP

Theorem 4.7 $^{\exists}\delta\text{-BPP} = {}^{\exists}\delta\text{-PP} = \mathsf{PSPACE}$ for $0 < \delta < \frac{1}{2}$.

Proof. It is sufficient to show that $\mathsf{ABPP} \subseteq {}^{\exists}\delta\text{-}\mathsf{BPP}$ for $0 < \delta < \frac{1}{2}$ from Theorem 2.9 together with the fact that ${}^{\exists}\delta\text{-}\mathsf{BPP} \subseteq {}^{\exists}\delta\text{-}\mathsf{PP} \subseteq \mathsf{PSPACE}$. (We remark that ${}^{\exists}\delta\text{-}\mathsf{PP} \subseteq \mathsf{PSPACE}$ is easy to show since we assume that δ is a *rational* number. We will consider the case with an *irrational* bias in subsection 4.6.2.) Let M_3 be a STM such that $L(M_3) \in \mathsf{ABPP}$, and $2p(n)$ be the length of the computation of M_3 on input x of length n. Without loss of generality, we can assume that M_3 accepts with probability greater than $\frac{63}{64}$ for $x \in L$, and less than $\frac{1}{64}$ for $x \notin L$. On the computation of M_3, a *state pair* contains a nondeterministic state and the next two probabilistic states. That is, a computation of M_3 contains $p(n)$ state pairs.

Here, the PTM M_2' constructed in the proof of Theorem 4.6 simulates nondeterministic choices with a specific δ-random source. On the other hand, the PTM N in the proof of [Pap94, Theorem 11.4] simulates probabilistic choices with an arbitrary δ-random source. Combining M_2' and N, we construct a PTM M_3' which simulates M_3 with a specific δ-random source. Let $q(n) = \lceil -\frac{\log(8(p(n)+1))}{\log(2\sqrt{\delta(1-\delta)})} \rceil$, and $r(n) = \lceil \frac{3\log p(n)+6}{2\delta-2\delta^2} \rceil$. A sequence of $r(n)$ bits will be called *block*. The $2^{r(n)}$ possible blocks are denoted by the corresponding binary integers $0, 1, \cdots, 2^{r(n)}-1$. If $\kappa = (\kappa_1, \kappa_2, \cdots, \kappa_{r(n)})$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_{r(n)})$ are blocks, then their *inner product* is defined $\kappa \cdot \lambda = \sum_{i=1}^{r(n)} \kappa_i \lambda_i \pmod 2$. Notice that the inner product of two blocks is a bit. M_3' simulates $2^{r(n)}$ M_3s in parallel to simulate probabilistic choices. The jth state pair of ith simulation of an M_3 is performed by the following:

(Simulation of a nondeterministic choice:)

(1) when M_3 nondeterministically selects m-choice ($m = 0$ or 1), assign $(1-\delta)$ to the probability that the outcome of a coin tossing is m; and
(2) select k-choice, where k is the majority of the outcomes of $q(n)$ coin tossing.

(Simulation of one of two probabilistic choices:)

(3) generate β_j with $r(n)$ bits generated by a δ-random source; and
(4) select $h_{(i,j)}$-choice, where $h_{(i,j)} = \beta_j \cdot i$;

At the end of the simulation, M_3' accepts if at least half of the $2^{r(n)}$ simulations accepts, or rejects otherwise.

When $x \in L$, as in the proof of Theorem 4.6, M_3' successes in simulating $p(n)$ nondeterministic choices with probability greater than $e^{-1/8}$. In the case, using the technique in the proof of Theorem 11.4 in [Pap94], we can show that the majority of M_3s rejects with probability less than $\frac{1}{8}$. Thus M_3' accepts with probability greater than $e^{-1/8} \cdot \frac{7}{8} > \frac{3}{4}$. On the other hand, when $x \notin L$, using the technique in the proof of Theorem 11.4 in [Pap94], we can show that the majority of M_3s accepts with probability less than $\frac{1}{8}$ for every nondeterministic choices. Consequently, M_3' rejects with probability greater than $\frac{3}{4}$. Thus $L \in {}^{\exists}\delta\text{-}\mathsf{BPP}$. ∎

In the proof, notice that we can't replace a simulation of a probabilistic choice by a fair coin tossing. Although fair coin tossing works for $x \in L$, it does not work for $x \notin L$, since a probabilistic choice must be simulated by *every* δ-random source.

We summarize the results in sections 4.2 through 4.4 in Figure 4.1.

4.5 The class ARP

In the previous section, we have indirectly shown that ${}^{\exists}\delta\text{-}\mathsf{BPP} \subseteq \mathsf{PSPACE}(= \mathsf{ABPP})$. Moreover, in our model, it is sufficient to consider the δ-random sources that only assign one of the probabilities δ and $1-\delta$ from Proposition 2.24. Thus, we can directly show that ${}^{\exists}\delta\text{-}\mathsf{BPP} \subseteq \mathsf{ABPP}$ and ${}^{\exists}\delta\text{-}\mathsf{PP} \subseteq \mathsf{APP}$ using the gadget in Figure 4.2. Although the gadget in the figure only works for the δ-random sources with $\delta = \frac{1}{4}$, Knuth and Yao showed that for an arbitrarily rational

$$
\begin{array}{ccccccccccc}
\mathsf{P} & = & 0\text{-}\mathsf{RP} & = & 0\text{-}\mathsf{BPP} & = & 0\text{-}\mathsf{PP} & & \\
 & & \bigcap\vert & & \bigcap\vert & & \bigcap\vert & & \\
 & & \delta\text{-}\mathsf{RP} & \subseteq & \delta\text{-}\mathsf{BPP} & = & \delta\text{-}\mathsf{PP} & & \\
 & & \| & & \| & & \bigcap\vert & & \\
 & & \frac{1}{2}\text{-}\mathsf{RP} & & \frac{1}{2}\text{-}\mathsf{BPP} & & \frac{1}{2}\text{-}\mathsf{PP} & & \\
 & & \| & & \| & & \| & & \\
 & & \mathsf{RP} & \subseteq & \mathsf{BPP} & \subseteq & \mathsf{PP} & & \\
 & & \| & & \| & & \| & & \\
 & & {}^{\exists}\frac{1}{2}\text{-}\mathsf{RP} & & {}^{\exists}\frac{1}{2}\text{-}\mathsf{BPP} & & {}^{\exists}\frac{1}{2}\text{-}\mathsf{PP} & & \\
 & & \bigcap\vert & & & & & & \\
 & & {}^{\exists}\delta\text{-}\mathsf{RP} & \subseteq & {}^{\exists}\delta\text{-}\mathsf{BPP} & = & {}^{\exists}\delta\text{-}\mathsf{PP} & = & \mathsf{PSPACE} \\
 & & \| & & & & & & \\
\mathsf{NP} & = & {}^{\exists}0\text{-}\mathsf{RP} & = & {}^{\exists}0\text{-}\mathsf{BPP} & = & {}^{\exists}0\text{-}\mathsf{PP} & &
\end{array}
$$

Figure 4.1: Summary of the results ($0 < \delta < \frac{1}{2}$)

number δ, we can construct such a gadget [KY76]. From this observation, we can show the following theorem.

Theorem 4.8 $\mathsf{ARP} = \mathsf{NP}$.

Proof. It is easy to see that $\mathsf{ARP} \subseteq \mathsf{NP}$ holds. Using the gadget shown in Figure 4.2 together with Theorem 4.6, we can show that $\mathsf{NP} = {}^{\exists}\frac{1}{4}\text{-}\mathsf{RP} \subseteq \mathsf{ARP}$. This establishes the theorem. ∎

4.6 Remarks

4.6.1 Characterization of a worst tree

The structure of a clustered tree plays an important role as the structure of a worst tree in the sections 4.1 and 4.2. Then, do clustered trees completely characterize worst trees? The answer is no.

Definition 4.9 Let T be a δ-tree of depth $d \geq 0$. T is a *scattered tree* if

(1) $d = 0$, or
(2) $0 \leq |T_r| - |T_l| \leq 1$, and both T_r and T_l are scattered trees.

Intuitively, a scattered tree is a tree whose all subtrees of the same depth have the same number of accepting leaves. Although the structure of a scattered tree is opposite to the structure of a clustered tree, the following corollary holds.

Corollary 4.10 Every scattered tree is worst.

Proof. By Corollary 4.4, it is sufficient to show that for any scattered tree T of depth d, $\Pr[T|F_{\max}(T)] = \sum_{i=0}^{k} \delta^i (1-\delta)^{d-t_{k-i}-i}$, where $t_0, t_1, \cdots, t_k$ are the integers such that $|T| = \sum_{i=0}^{k} 2^{t_i}$ with $t_k > \cdots > t_1 > t_0 \geq 0$. An induction on the depth of the tree shows the fact. ∎

Moreover, combining the definitions of clustered and scattered trees, we can define a *chimera tree* as follows:

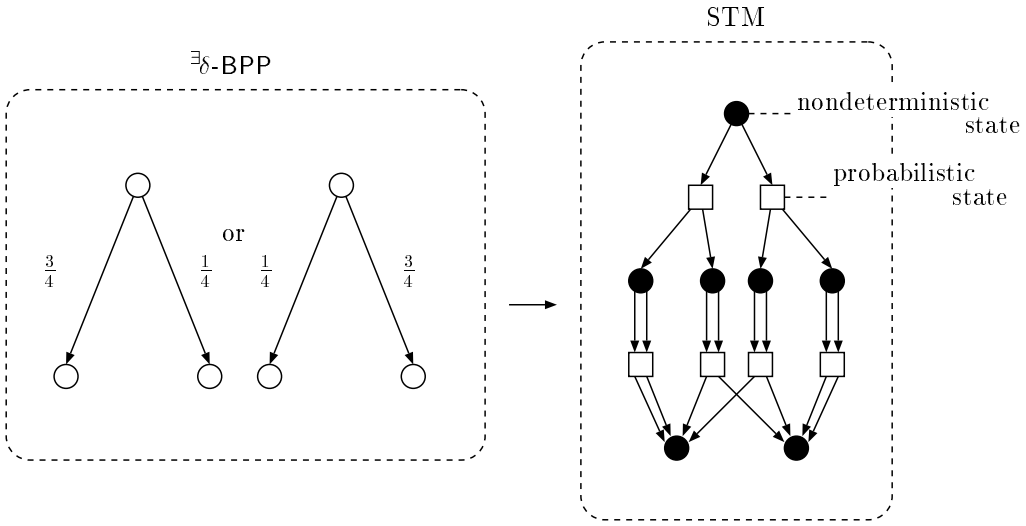

Figure 4.2: gadget for δ-random sources with $\delta = \frac{1}{4}$

Definition 4.11 Let T be a δ-tree of depth $d \geq 0$. Then T is a *chimera tree* if
(1) $d = 0$,
(2) $|T_l| = 0$, and T_r is a chimera tree,
(3) T_l is a chimera tree and $|T_r| = 2^{d-1}$, or
(4) $0 \leq |T_r| - |T_l| \leq 1$ and both T_l and T_r are chimera trees.

By a simple induction on the depth of the tree together with Theorem 4.3, Corollaries 4.4 and 4.10, it is not hard to see that a chimera tree is worst also. Moreover, as shown in the proof of Lemma 4.2, a δ-tree has many subtrees that are exchangeable each other. Exchanging these subtrees of a chimera tree, we get many variations of a worst tree. Although the author conjectures that every worst tree is a chimera tree or its variation, the complete characterization of a worst tree remains an open problem.

4.6.2 Complexity classes with irrationally δ-random sources

In the sections above, we only consider a δ-random source with rational δ. All results also can be shown for a δ-random source with irrational δ except Theorem 4.7. That is, $^{\exists}\delta$-PP $\subseteq$ PSPACE is not obvious when δ is an *irrational* number. How can we simulate a PTM with an irrational δ-random source by a DTM with polynomial space? In this subsection, we solve this problem. We first show the following lemma to deal with an *arbitrary* δ.

Lemma 4.12 Let L be a language with $L \in {}^{\exists}\delta$-PP for some δ. Then there exists a number δ' such that $L \in {}^{\exists}\delta'$-PP and δ' can be represented in polynomial space for the input length.

Proof. Let L be a language with $L \in {}^{\exists}\delta$-PP for some δ, and M_4 be a PTM such that $L(M_4) = L$. Let p be the length of a computation of M_4. (We write only p, depending on the input length, for short.) Let $d = \frac{\delta^p}{2^{p+1}p(1-\delta)^{p-1}}$. Since d can be represented in polynomial space for the input length, there exists some δ' with $|\delta' - \delta| < d$ such that δ' can be represented in polynomial space for the input length. Without loss of generality, we assume that $\delta' > \delta$. We simulate

M_4 in polynomial space by substituting δ' for δ. Then the error of the probability of a leaf by substituting is equal to $\delta^i(1-\delta)^{p-i} - \delta'^i(1-\delta')^{p-i}$ for some i with $0 \leq i \leq p$. If the sum of every error of the probability of the leaf is less than half of the probability of any leaf, $L(M_4)$ is also in ${}^{\exists}\delta'$-PP. Here, the minimum probability of a leaf is equal to δ^p. On the other hand, an error of the probability of a leaf is at most $\max\{\delta'^p - \delta^p, (1-\delta)^p - (1-\delta')^p\} = (1-\delta)^p - (1-\delta')^p$ since $\delta' > \delta$. Thus, since M_4 has 2^p leaves, the sum of every error is at most

$$2^p \ |\delta^i(1-\delta)^{p-i} - \delta'^i(1-\delta')^{p-i}| < 2^p \left((1-\delta)^p - (1-\delta')^p\right).$$

Using Taylor series, we get

$$2^p \left((1-\delta)^p - (1-\delta')^p\right) < 2^p pd(1-\delta)^{p-1} = \frac{\delta^p}{2}.$$

Hence, the sum of every error of the probability of any computation tree is less than a half of the probability of any leaf. This implies the lemma. ∎

Lemma 4.13 For an *arbitrary* δ with $0 < \delta < \frac{1}{2}$, ${}^{\exists}\delta$-PP $\subseteq$ PSPACE.

Proof. Let L be a language with $L \in {}^{\exists}\delta$-PP for some δ. Let M_5 be a PTM, such that $L = L(M_5)$. Let δ' be a number satisfying the conditions in the proof of Lemma 4.12. We construct an NTM M_5', which accepts L as follows;

1: nondeterministically compute δ';
2: simulate all computations of M_5, and counts up its probability by using δ' instead of δ; and
3: accept if the probability is greater than $\frac{1}{2}$; or reject otherwise.

Clearly, M_5' uses at most polynomial space for the input length, and $L = L(M_5')$. Thus $L \in$ PSPACE. ∎