Chapter 7

NP-completeness of the problems on a simple structure

7.1 NP-completeness of Hi-Q

In this section, we consider a puzzle commonly known as Hi-Q. The board of the puzzle consists of 33 points arranged as in Figure 7.1 with all but one point placed with tokens. This special point, the center of the board, is named the *goal*. A move consists of jumping one token over an adjacent one onto an empty point. The jumping must be made either horizontally or vertically. When a token is jumped it is removed from the board. The objective of the puzzle is to make the board to a position with just one token on the goal. Thus a sequence of jumps which leads the board consisting of no tokens but on the goal will be an answer for the puzzle.

The puzzle is generalized in such a way that an initial position is given on the extended board of size $n \times n$, and a goal is also given on which only one token will finally be placed. Now call the puzzle *generalized Hi-Q*. The theorem in this section is the following:

Theorem 7.1 The question: Given generalized Hi-Q, is there an answer for the generalized puzzle? is NP-complete.

Proof. It is easy to see that the decision problem belongs to NP. It then remains to prove the completeness. We provide a polynomial time reducibility from the following known NP-complete problem: Hamiltonian Cycle Problem in planar digraph with degree bound two [Ple79].

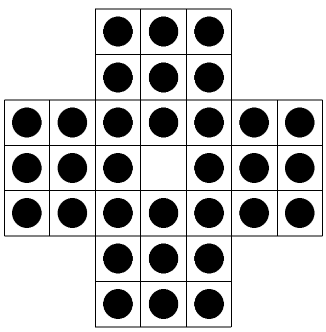


Figure 7.1: An initial board of Hi-Q

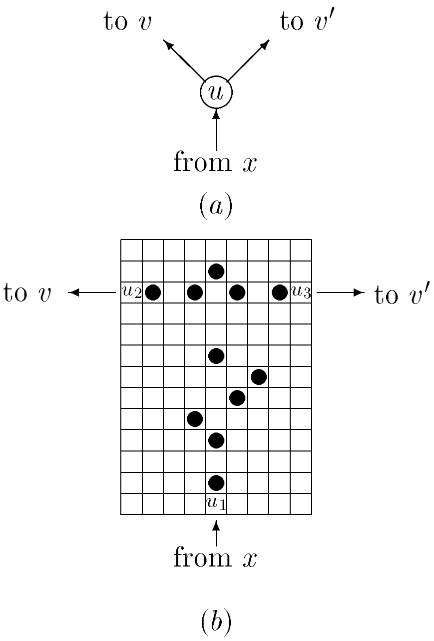


Figure 7.2: A type $\bigtriangledown$ node u of indegree one and outdegree two

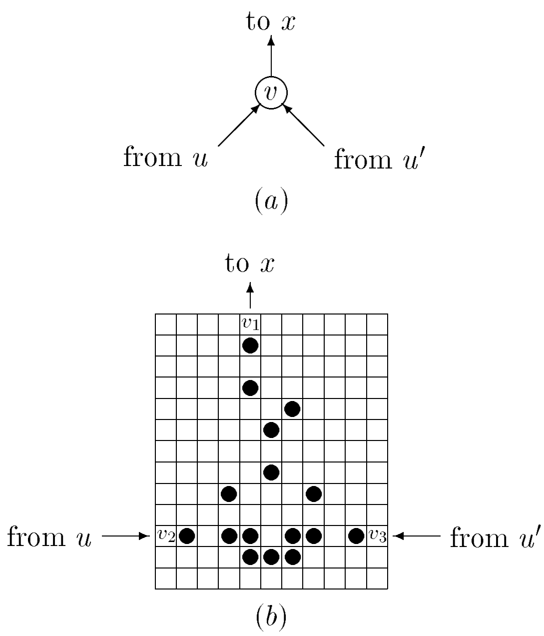


Figure 7.3: A type $\bigtriangleup$ node v of indegree two and outdegree one

Let $\mathcal{D}$ be the class of planar digraphs where each node is either indegree one and outdegree two, called type $\bigtriangledown$ (Figure 7.2(a)), or indegree two and outdegree one, called type $\bigtriangleup$ (Figure 7.3(a)).

Hamiltonian Cycle Problem of $\mathcal{D}$ is

Given: an element D of $\mathcal{D}$.

Question: Does D contain a Hamiltonian cycle?

Let D constitute a given instance of Hamiltonian Cycle Problem in planar digraph with degree bound two. We construct a corresponding instance of generalized Hi-Q. The board of the generalized puzzle will be constructed as follows:

1. For each node u of type $\bigtriangledown$, construct a pattern of token placement shown in Figure 7.2(b) and call it the *pattern* u. The points u_1 and u_2 are said to be the *exits* and u_3 be the *entry* of the pattern u. For each node v of type $\bigtriangleup$, construct similarly a pattern v as in Figure 7.3(b). The points v_2 and v_3 are the *entries* and the point v_1 is the *exit* of the pattern v.

2. For each edge (u, v), connect two patterns corresponding to u and v between the exit of u and the entry of v, using extension modules shown in Figure 7.4. Call c_1 the entry, and c_2 the exit in Figure 7.4. Two extension modules can be connected by identifying the exit of one extension module with the entry of another module. An extension module and a pattern of Figure 7.2(b) or 7.3(b) can be similarly connected.

Note that since D is planar, we can put patterns on the board so that the connection between them by the procedure 2 above may not cross.

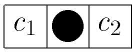
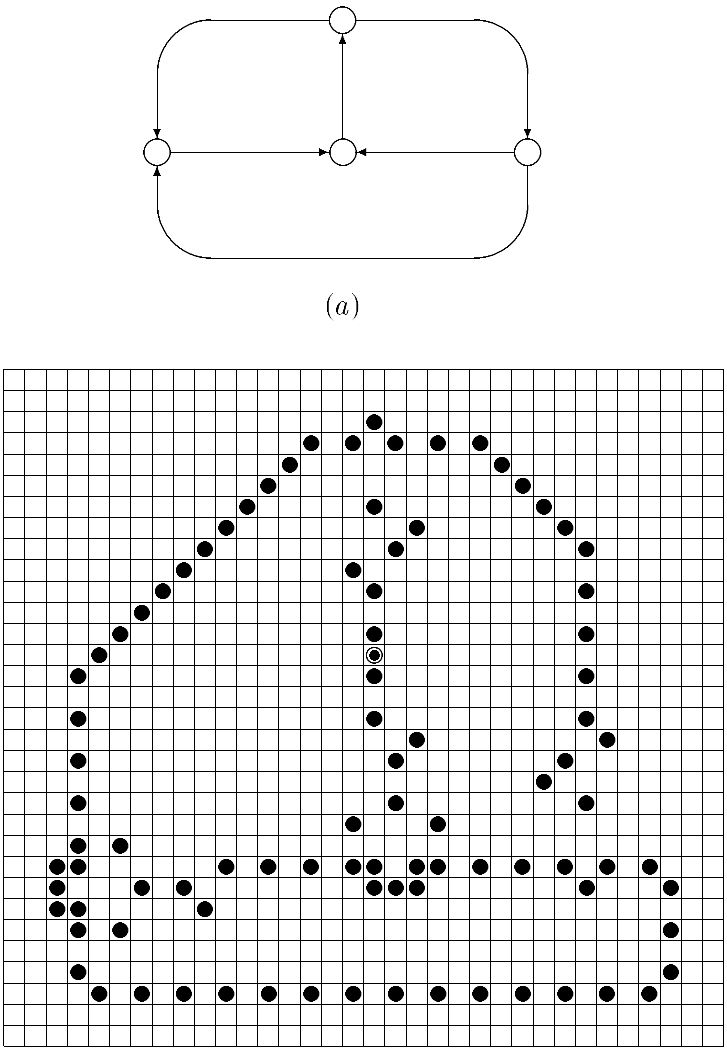


Figure 7.4: An extension module



(a)

(b)

Figure 7.5: An example of the construction

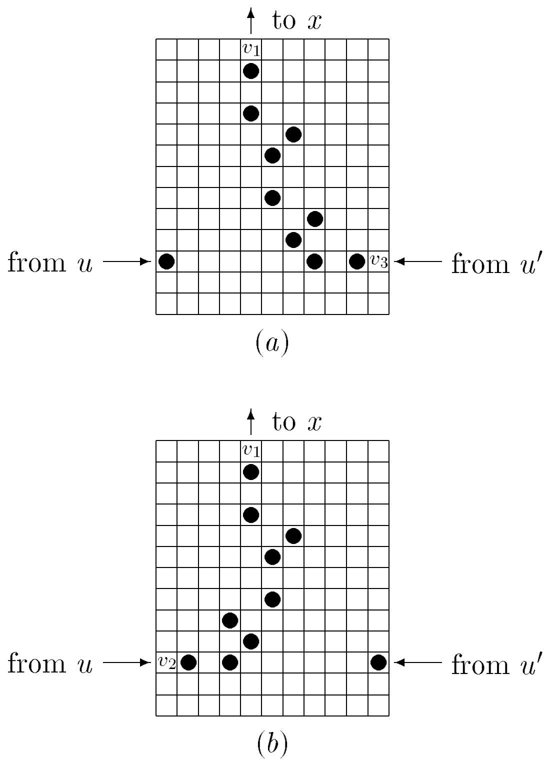


Figure 7.6: Placement after six jumps from Figure 7.3(b)

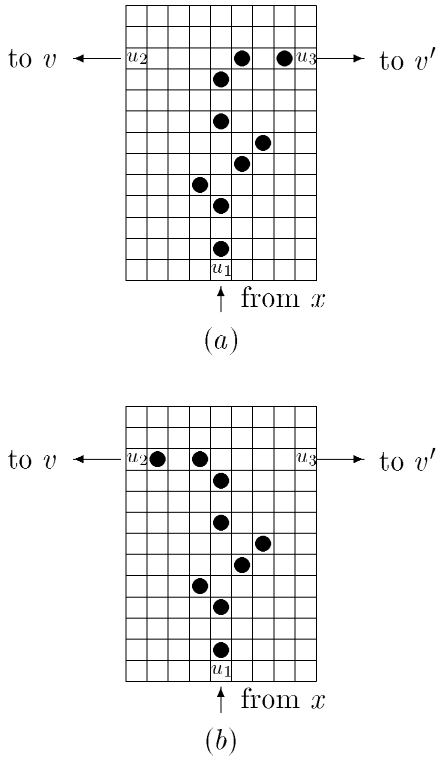


Figure 7.7: Placement after three jumps from Figure 7.2(b) with additional token on u_2 or u_3

Select one node of type $\bigtriangledown$, say z, and put a token on the entry z_1 of the pattern z. The point z_1 is the goal of the constructed generalized Hi-Q.

As an example, for a digraph shown in Figure 7.5(a), the constructed puzzle is shown in Figure 7.5(b) where a token surrounded by a circle denotes the goal.

Now we will show that D has a Hamiltonian cycle if and only if there is an answer in the constructed puzzle.

Assume that $D = (V, A)$ has a Hamiltonian cycle. For each edge $\tilde{e} = (\tilde{u}, \tilde{v})$ not on the Hamiltonian cycle, we first remove the tokens related to $\tilde{e}$ from the board: we note that $\tilde{u}$ should be of type $\bigtriangledown$ and $\tilde{v}$ be of type $\triangle$. Hence, the edge $\tilde{e} = (\tilde{u}, \tilde{v})$ should be either (u, v) or (u, v') in Figure 7.2(a), and be either (u, v) or (u', v) in Figure 7.3(a). Now suppose that $e = (u, v)$ is not on the Hamiltonian cycle, and that u, v are nodes as shown in Figure 7.2(a) and Figure 7.3(a), respectively. First, we make six jumps from Figure 7.3(b) to place a token on the entry v_2 from u as in Figure 7.6(a). (If e is of the form (u', v), a token is placed on the entry v_3 from u' as in Figure 7.6(b).) Next, the token on the entry jumps to remove the tokens on the extension modules from u to v, and it moves to the exit u_2 in the placement of Figure 7.2(b); with the token on the exit, we make three moves and the placement can be the one in Figure 7.7(a). (If e is of the form (u, v'), a token is placed on the exit u_3 to v', and then become the one in Figure 7.7(b).) After executing this procedure for every edge not on the Hamiltonian cycle, we note that all the placements of the forms Figures 7.2(b) and 7.3(b) would be the ones shown in Figure 7.7(a) or 7.7(b), and Figure 7.6(a) or 7.6(b), respectively, and that the tokens which remain on extension modules are of the edges belonging to the Hamiltonian cycle.

Then the token on the goal begins to jump to remove the tokens on the board along the direction of the edges in the Hamiltonian cycle, and finally one token can remain on the goal.

Assume that there is an answer in the constructed puzzle. We show that there is a Hamiltonian cycle in D. In order to prove this, we need some lemmas:

Lemma 7.2 Consider a placement of tokens constructed for nodes u and v shown in Figure 7.2(b) and Figure 7.3(b), respectively. We say that u_i and v_i, $1 \leq i \leq 3$ is a *gate*. Let $S \subseteq \{u_1, u_2, u_3\}$, and $T \subseteq \{v_1, v_2, v_3\}$. With an additional token on each gate of S (T), assume that every token on the figure is removed by a series of jumps, and that a token remains on each gate $\{u_1, u_2, u_3\} - S$ ($\{v_1, v_2, v_3\} - T$, respectively) thereafter. Then $S = \{u_1, u_2\}$ or $S = \{u_1, u_3\}$, and $T = \{v_2\}$ or $T = \{v_3\}$.

Proof. We first show that $T = \{v_2\}$ or $T = \{v_3\}$. It is obvious that $T \neq \phi$ and $v_1 \notin T$ from the construction of the placement of Figure 7.3(b). For $T = \{v_2, v_3\}$, we have examined every possible sequence of jumps by using a computer from the placement of Figure 7.3(b) with two additional tokens on v_2 and v_3, and found that no matter how tokens jump, at least one token remains on non-gate point in the figure. Thus, $T \neq \{v_2, v_3\}$.

We now show the proof on S. By the construction of the placement of Figure 7.2(b), we obtain that $u_1 \in S, S \neq \{u_1, u_2, u_3\}$, and $S \neq \{u_1\}$. Then the lemma follows. ∎

Remark. We note that for $T = \{v_2\}$ ($T = \{v_3\}$) in the placement of Figure 7.3(b), there are two possible independent moves which make all tokens of the placement removed: (1) a move for a token to jump to v_3 (v_2), and (2) having an additional token on v_2 (v_3, respectively), a move for a token to come up on v_1.

We say that a series of jumps is an *edge-directed move* if it removes tokens of placement of Figure 7.3(b) or Figure 7.2(b) in the direction from an entry to an exit, or if it removes tokens of extension modules in the same direction as the edge in D, where the extension modules are constructed from.

Lemma 7.3 There is exactly one edge-directed move in the constructed puzzle during the sequence of jumps to leave only one token on the goal.

Proof. Assume that an edge-directed move passes a placement of the form Figure 7.2(b) or Figure 7.3(b). Then Lemma 1 insists that after the passage there is still one edge-directed move. At the beginning of the puzzle, there is an edge-directed move starting from the goal. Therefore the lemma follows. ∎

We now proceed with the proof of the theorem. The existence of the sequence of jumps to remove all tokens but one on the goal implies that the edge-directed move visits every placement of Figures 7.2(b) and 7.3(b) one after another. Also the edge-directed move starts from the goal and ends at the goal. We have selected the node z of D of the form Figure 7.2(a) to choose z_1 as the goal. Now we can make a Hamiltonian cycle in D starting and ending at z by traversing corresponding nodes in order of placements that the edge-directed move visits. ∎

7.2 NP-complete problems on a 3-connected cubic planar graph

In this section, we show NP-completeness of the vertex cover (VC) problem on a 3-connected cubic planar graph of girth greater than 3. To this end, we first show the following lemma for a gadget shown in Figure 7.8(a).

Lemma 7.4 Let $U = \{v_1, v_2, v_3, v_4\}$. Then

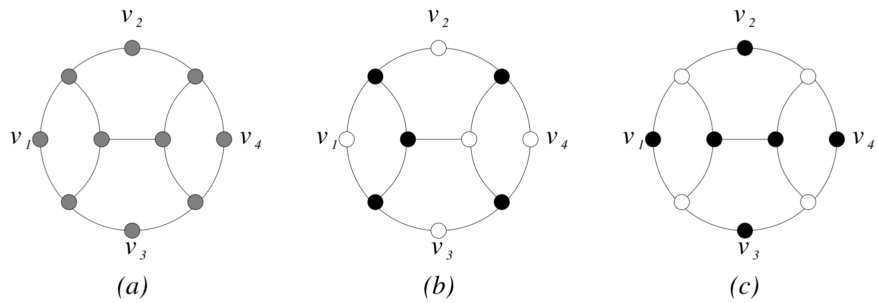


Figure 7.8: Gadget and its VCs

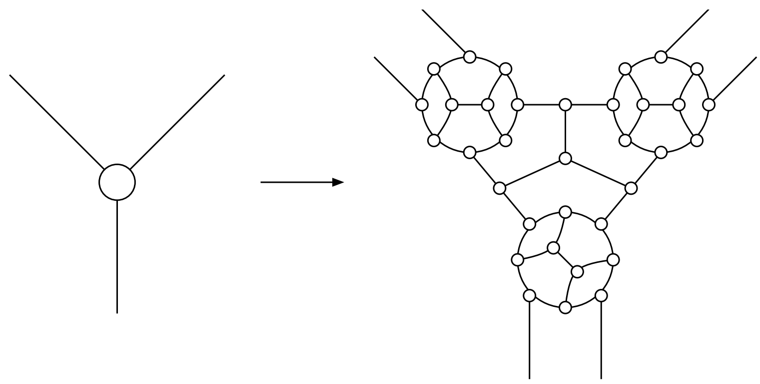


Figure 7.9: Replacement of a vertex by three gadgets

(1) the minimum VC of the gadget has size equal to 5, and it contains no vertex in U;
(2) for every VC on the gadget, if it contains at least one vertex in U, it has size greater than 5, and
(3) there exists a VC of size equal to 6, which contains every vertex in U.

Proof. We must select at least four vertices to cover eight edges around the gadget, and at least one of two center vertices to cover center edge. Thus every VC contains at least five vertices. Thus, the VC of size five is only one type shown in Figure 7.8(b) with black vertices, which contains no vertex in U. Now, it is not difficult to see that if a VC contains at least one vertex in U, it has size greater than 5. The VC shown in Figure 7.8(c) with black vertices completes the lemma. ∎

Now, we are ready to show main theorem in this section.

Theorem 7.5 The following problem is NP-complete: For a given 3-connected cubic planar graph G of girth greater than 3 and a positive integer k, is there a VC of size greater than k?

Proof. We provide a polynomial time reduction from the following known NP-complete problem [GJ79]: For a given cubic planar graph $G_0 = (V_0, E_0)$ and positive integer k_0, is there a VC of size greater than k_0? We replace all vertices of G_0 in parallel, each by three gadgets and four additional vertices, as depicted in Figure 7.9. Let $G = (V, E)$ be the resulting graph. Clearly, G is a 3-line-connected cubic planer graph with girth 4. Here, it is well-known that a cubic graph is 3-connected if and only if it is 3-line-connected (see e.g. [BM76]). Thus, G is also 3-connected. From Lemma 7.4, it is not difficult to see that G_0 has a VC of size k_0 if and only if G has a VC of size $k_0 + 18 \mid n \mid$. ∎

For any graph $G = (V, E)$, $V' \subseteq V$ is a maximum independent set of G if and only if $V - V'$ is a VC of G [GJ79]. This implies the following corollary.

Corollary 7.6 The following problem is NP-complete: For a given 3-connected cubic planar graph G of girth greater than 3 and a positive integer k, is there a maximum independent set of size k?

Das and Goodrich proved NP-completeness of the following two problems; the VC problem on a 3-connected planar graphs, and the VC problem on a c-stellations of a 3-connected cubic planar graph for any constant $c \geq 4$. Here, a *stellation* of a face f in G as the insertion of a vertex in the interior of f that we then make adjacently to each vertex of f. A c-stellation of a graph is the result of performing c consecutive stellations on the graph (see [DG95] for detail). These VC problems contain the VC problem in Theorem 7.5. Thus, using the theorem, we can simplify the proofs in [DG95]. We remark that the theorem also simplify the proof of NP-completeness of the problem computing the asteroidal number of a graph (see [KKM96]).

7.3 NP-complete problems on a triangulation of the sphere

Whitney expressed planarity with the existence of dual graphs. Given a plane graph G, its *geometric dual graph* G^* is constructed as follows; place a vertex in each face of G (including the exterior region) and, if two regions have an edge e in common, join the corresponding vertices by an edge e^* crossing only e. The geometric dual graph of a 3-connected graph is always a simple graph (see [Har72] for detail). A *triangulation* of the sphere is a planar graph such that every face of the graph, including the exterior region, is a triangle. We first show the following lemma.

Lemma 7.7 Geometric dual graph of a 3-connected cubic planar graph is a triangulation of the sphere.

Proof. Let $G = (V, E)$ be a 3-connected cubic planar graph and $G^* = (V^*, E^*)$ be its geometric dual graph. Since G is cubic, $3 \mid V \mid = 2 \mid E \mid$ holds. By definition of the duality, $\mid E \mid = \mid E^* \mid$ holds. Since G is connected, Euler's Theorem states that $2+ \mid E \mid - \mid V \mid - \mid V^* \mid = 0$ (see [Koz92, Lecture 14] for example). Combining the equations, we get that $\mid E^* \mid = 3 \mid V^* \mid -6$. Here, G^* is a triangulation of the sphere if and only if $\mid E^* \mid = 3 \mid V^* \mid -6$ (see [Koz92, Theorem 14.10] for example). This completes the proof. ∎

For a given triangulation of the sphere and a positive integer k, we define two problems on them as follows.

Independent triangle set problem:

Question: Are there a triangle set of size k or more such that no two triangles in the set have an edge in common?

Shaping triangulation set problem:

Question: Are there a triangle set of size k or less such that every edge of the triangulation is had at least one triangle in the set?

We remark that the shaping triangulation problem is the problem to minimize the number of triangles to shape the triangulation of the sphere.

Theorem 7.8 Both the independent triangle set problem and the shaping triangulation problem are NP-complete.

Proof. By Lemma 7.7, the maximum independent set problem and the VC problem on a 3-connected cubic planar graph can be reduced to the independent triangle set problem, and the shaping triangulation problem, respectively. ∎

It is easy to see that the girth of a graph corresponds to the minimum degree of its geometric dual graph. Thus, the independent triangle set problem and the shaping triangulation problem remain NP-complete even if we add the restriction that the minimum degree is greater than 3 to a given triangulation.