Collapse of PP with a Semi-Random Source to BPP

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1 Introduction

Satha and Vazirani introduced a *semi-random source* as an extremely general model of an imperfect source of randomness [SV84, SV86]. In a semi-random source, the previous bits generated by the source can condition the next bit in an arbitrarily way. Accordingly, the next bit is generated by the flip of a coin whose bias is fixed by an adversary who has complete knowledge of the history of the process. The adversary is limited to choosing a bias in $[\delta, 1-\delta]$ with some positive number $0 \leq \delta \leq \frac{1}{2}$. We call a semi-random source with parameter δ a *δ-random source*.

We first characterize the probability of the computation tree of a PTM having access to a semi-random source rather than a perfect random source. The probability that such a PTM accepts depends on the probability of each choice assigned by a given semi-random source. We assume that the probability that a PTM with a semi-random source accepts is maximized. In this case, do two computation trees having the same number of accepting paths have the same accepting probability? The answer is no. The accepting probability depends not only on the number of accepting paths, but also on the distribution of the accepting paths in the computation tree of the PTM. We define the *best* and *worst trees* that give the maximum and minimum values, respectively. The main result is that we find the values. The maximum probability of a best tree of depth d with t accepting paths with a δ-random source is $\sum_{i=0}^{t_0} \binom{d}{i} \delta^i (1-\delta)^{d-i} + t_1 \delta^{t_0+1} (1-\delta)^{d-t_0-1}$, where $t = \sum_{i=0}^{t_0} \binom{d}{i} + t_1$ with $t_1 \leq \binom{d}{t_0+1}$. The maximum probability of a worst tree of depth d with t accepting paths with a δ-random source is $\sum_{i=0}^{k} \delta^i (1-\delta)^{d-t_{k-i}-i}$, where $t = \sum_{i=0}^{k} 2^{t_i}$ and $t_k > \cdots > t_1 > t_0 \geq 0$.

Gill defined the classes, PP, BPP and RP by PTMs that have access to a perfect random source [Gil77]. What if the PTMs only have access to a semi-random source rather than a perfect random source? This question was first considered by Vazirani and Vazirani [VV85], who defined the class SR_p corresponding to RP, and showed that $SR_p = \mathsf{RP}$. Later, Vazirani similarly defined the class $SBPP$ corresponding to BPP, and prove that $SBPP = \mathsf{BPP}$. In the same manner, we introduce the class δ-PP corresponding to PP. Using the result for the maximum probability of a worst tree, we prove that δ-PP = BPP for $0 < \delta < \frac{1}{2}$. Compared with that $SR_p = \mathsf{RP}$ and $SBPP = \mathsf{BPP}$, this result is interesting.

We finally consider the structure of a worst tree, which plays an important role in this paper. Compared with a best tree, the structure of a worst tree has many variations. The complete characterization of a worst tree remains an open problem.

2 Preliminaries

We assume a standard Turing machine model. A *probabilistic Turing machine* (PTM) is a Turing machine with distinguished states called coin-tossing states. For formal definition of a PTM, see [Gil77, BDG95]. Without loss of generality, we assume that a PTM is standardized as follows: It is a polynomial time bounded PTM with exactly two choices per step. Therefore, its computation tree is a full binary tree whose depth is polynomial in the length of the input.

A semi-random source with parameter δ, or simply a δ-random source, is defined as follows.

Definition 1 [SV84] *Let δ be a number such that $0 \leq \delta \leq \frac{1}{2}$. A* semi-random source *with parameter δ outputs bits $X_1X_2\cdots$, such that for all i and for all $x_1, x_2, \cdots$,*

$$\delta \leq \Pr[X_i = x_i \,|\, X_1 = x_1, \cdots, X_{i-1} = x_{i-1}\,] \leq 1 - \delta.$$

Definition 2 [VV85] *A language L is in δ-RP if there exists a PTM M such that (1) for $x \in L$, M accepts with probability greater than $\frac{1}{2}$ for all δ-random sources and (2) for $x \notin L$, M always rejects.*

Definition 3 [Vaz86] *A language L is in δ-BPP if there exists a PTM M such that (1) for $x \in L$, M accepts with probability greater than $\frac{3}{4}$ for all δ-random sources and (2) for $x \notin L$, M accepts with probability less than $\frac{1}{4}$ for all δ-random sources.*

We note that the class δ-RP is denoted by SR_p in [VV85], and the class δ-BPP is denoted by $SBPP$ in [Vaz86]. The notations in this paper follow from [Pap94].

Definition 4 *A language L is in δ-PP if there exists a PTM M such that (1) for $x \in L$, M accepts with probability greater than $\frac{1}{2}$ for all δ-random sources and (2) for $x \notin L$, M accepts with probability less than $\frac{1}{2}$ for all δ-random sources.*

It is not hard to see that 0-RP = 0-BPP = 0-PP = P, $\frac{1}{2}$-RP = RP, $\frac{1}{2}$-BPP = BPP, and $\frac{1}{2}$-PP = PP (see [Pap94]). The following results are known.

Theorem 5 [VV85] *δ-RP = RP for $0 < \delta \leq \frac{1}{2}$.*

Theorem 6 [Vaz86] *δ-BPP = BPP for $0 < \delta \leq \frac{1}{2}$.*

To consider the distribution of the accepting paths in the computation tree of a PTM, we define some notations. A *δ-tree* is a rooted full binary tree whose every leaf is labeled "accept" or "reject". The leaves labeled "accept" and "reject" are respectively called *accepting* and *rejecting leaves*. For a δ-tree T, we denote by $|T|$ the number of accepting leaves of T.

Definition 7 *A* δ-assignment F *to a δ-tree is a mapping from the set of edges of the δ-tree to the interval $[\delta, 1-\delta]$ such that the two edges leaving each internal node are assigned values whose sum is 1.*

Definition 8 *Let T be a δ-tree, and F be a δ-assignment to T. The* probability of a node *of T under F is the product of the values assigned to the edges on the path from root to the node by F. The* probability of T *under F, denoted by $\Pr[T\,|\,F]$, is the sum of the probabilities of the accepting leaves.*

Definition 9 *For a given δ-tree T and an assignment F to T, F is* a maximum assignment *if $\Pr[T\,|\,F] \geq \Pr[T\,|\,F']$ for every δ-assignment F' to T.*

Proposition 10 *For every δ-tree T, there exists a maximum assignment F to T such that F only assigns δ or $1-\delta$ to each edge of T.*

Proof. For a given δ-tree T, we recursively define an assignment F as follows: First, let v be a node whose sons are leaves. If at least one of its sons is accepting leaf, assign $(1-\delta)$ to the edge between v and the son, and assign δ to the other edge. Otherwise, assign $(1-\delta)$ and δ to the two edges between v and its two sons in any way. Second, let u be a node whose sons are the roots of two subtrees, say T_0 and T_1, whose all edges are already assigned. If $\Pr[T_0 \,|\, F] > \Pr[T_1 \,|\, F]$, assign $(1-\delta)$ to the edge between u and T_0 and assign δ to the edge between u and T_1. Otherwise, assign $(1-\delta)$ to the edge between u and T_1 and assign δ to the edge between u and T_0. Using a simple induction on the depth of the tree, we find that $\Pr[T \,|\, F] \geq \Pr[T \,|\, F']$ for every δ-assignment F' to T. ∎

By the proposition, it is sufficient to consider δ-assignments that only assign the values δ and $1-\delta$ to maximize the probability. For a given δ-tree T, we denote by $F_{\max}(T)$ the maximum assignment defined in the proof of the proposition.

Definition 11 *A δ-tree T_w is* worst *if $\Pr[T_w \,|\, F_{\max}(T_w)] \leq \Pr[T \,|\, F_{\max}(T)]$ for every δ-tree T with $|T_w| = |T|$. A δ-tree T_b is* best *if $\Pr[T_b \,|\, F_{\max}(T_b)] \geq \Pr[T \,|\, F_{\max}(T)]$ for every δ-tree T with $|T_b| = |T|$.*

Proposition 12 *Let T_w be a worst tree and T_b be a best tree such that $\Pr[T_w \,|\, F_{\max}(T_w)] = \Pr[T_b \,|\, F_{\max}(T_b)]$. Then, for every δ-tree T such that $\Pr[T_w \,|\, F_{\max}(T_w)] = \Pr[T \,|\, F_{\max}(T)]$, $|T_w| \geq |T| \geq |T_b|$.*

Proof. There exists a worst tree T' such that $\Pr[T' \,|\, F_{\max}(T')] \leq \Pr[T \,|\, F_{\max}(T)] = \Pr[T_w \,|\, F_{\max}(T_w)]$ with $|T'| = |T|$. Thus, $|T_w| \geq |T'| = |T|$. A symmetric argument works for the best tree. ∎

Proposition 13 *Let T_b be a best tree of depth d with a δ-random source. Then, $\Pr[T_b \,|\, F_{\max}(T_b)] = \sum_{i=0}^{t_0} \binom{d}{i} \delta^i (1-\delta)^{d-i} + t_1 \delta^{t_0+1} (1-\delta)^{d-t_0-1}$, where $|T_b| = \sum_{i=0}^{t_0} \binom{d}{i} + t_1$ with $t_1 \leq \binom{d}{t_0+1}$.*

Proof. For each i with $0 \leq i \leq d$, a δ-tree of depth d has $\binom{d}{i}$ leaves corresponding to the probability $\delta^i (1-\delta)^{d-i}$, and $\delta^i (1-\delta)^{d-i} < \delta^{i-1} (1-\delta)^{d-i+1}$ since $\delta < (1-\delta)$. Thus, taking $|T_b|$ leaves which correspond to the largest $|T_b|$ probabilities, and labeling them "accept", we have a best tree. This implies the proposition. ∎

The structure of a best tree is given by Proposition 13. Hereafter, we consider the structure of a worst tree. From now on, we assume that the two sons of each internal node in a δ-tree are respectively distinguished as the left and right sons. For given δ-tree T, the *left* and *right subtrees* of T are defined by the subtrees such that the roots of them are respectively the left and right son of the root of T. We denote by T_l and T_r the left and right subtrees of T, respectively. We similarly denote by T_{lr} the right subtree of T_l, and so on. We call the root of T the *parent* of T_l and T_r. We fix the assignment defined as follows; for each internal node, assign $(1-\delta)$ to every edge incident to a right son, and δ to every edge incident to a left son. The fixed assignment is the maximum assignment to the trees defined hereafter. It can be proved by simple inductions, and will be omitted.

Definition 14 *Let T be a δ-tree of depth $d \geq 0$. T is a* clustered tree *if (1) $d = 0$, (2) $|T_l| = 0$, and T_r is a clustered tree, or (3) T_l is a clustered tree, and $|T_r| = 2^{d-1}$.*

Intuitively, a clustered tree is a tree whose rightmost leaves are all accepting leaves.

3 Results

3.1 Probability of a worst tree

First, we show two crucial lemmas for a clustered tree.

Lemma 15 *Let T be a clustered tree of depth d. Let $t_0, t_1, \cdots, t_k$ be the integers such that $|T| = \sum_{i=0}^{k} 2^{t_i}$ with $t_k > \cdots > t_1 > t_0 \geq 0$. Then,* $\Pr[T \,|\, F_{\max}(T)] = \sum_{i=0}^{k} \delta^i (1-\delta)^{d-t_{k-i}-i}$.

Proof. From a δ-tree of depth d with no accepting leaf, we construct a δ-tree of depth d with $|T|$ accepting leaves using the following steps:

Step k: Let T_k be the rightmost subtree of depth t_k of the tree. Change all the labels of the leaves of T_k from "reject" to "accept".

Step i ($i = k-1, k-2, \cdots, 0$): Let T'_{i+1} be the subtree such that its parent is also the parent of T_{i+1} defined in step $i+1$. Let T_i be the rightmost subtree of depth t_i of T'_{i+1}. (Note that this step works since $t_{i+1} > t_i$.) Change all the labels of the leaves of T_i from "reject" to "accept".

Since each T_i is always taken from rightmost side, we obtain a clustered tree with $|T|$ accepting leaves after the construction. Thus the constructed tree is the same tree as T. The path from the root of T to the root of T_i consists of the path from the root of T to the parent of T_k (whose $(d - t_k - 1)$ edges are assigned $(1-\delta)$), an edge to the root of T'_k (which is assigned δ), the path to the parent of T_{k-1} (whose $(t_k - t_{k-1} - 1)$ edges are assigned $(1-\delta)$), an edge to the root of T'_{k-1}, $\cdots$, and the path to the root of T_i. Thus, the probability of the root of T_i ($k \geq i \geq 0$) is $\delta^i (1-\delta)^{d-t_{k-i}-i}$. Adding them together, we have the lemma. ∎

Lemma 16 *Let l, r and d be any integers with $0 \leq l \leq r \leq 2^{d-1}$, and T be a clustered tree of depth d with $|T| = l + r$. Let T' be a δ-tree of depth d such that $|T'_l| = l$, $|T'_r| = r$, and both T'_l and T'_r are clustered trees. Then,* $\Pr[T' \,|\, F_{\max}(T')] \geq \Pr[T \,|\, F_{\max}(T)]$.

Proof. We show the lemma by induction on the depth of the tree. Since it is clear when $d \leq 2$, we assume $d > 2$. Here, $\Pr[T' \,|\, F_{\max}(T')] = \delta^2 \Pr[T'_{ll} \,|\, F_{\max}(T')] + \delta(1-\delta)\Pr[T'_{lr} \,|\, F_{\max}(T')] + (1-\delta)\delta \Pr[T'_{rl} \,|\, F_{\max}(T')] + (1-\delta)^2 \Pr[T'_{rr} \,|\, F_{\max}(T')]$. Thus, the probability of T' does not change even if we exchange T'_{lr} and T'_{rl}. For T'_{ll} and T'_{rl}, four cases arise:
Case 1: $|T'_{ll}| > 0$ and $|T'_{rl}| = 0$. This case is impossible since $0 \leq l \leq r$.
Case 2: $|T'_{ll}| = |T'_{rl}| = 0$. Exchanging T'_{lr} and T'_{rl}, we find that only T'_r has accepting leaves with $|T'_{rl}| = l$ and $|T'_{rr}| = r$. Thus, by the inductive hypothesis, $\Pr[T'_r \,|\, F_{\max}(T'_r)] \geq \Pr[T_r \,|\, F_{\max}(T_r)]$. Since $|T_l| = |T'_l| = 0$, the lemma holds.
Case 3: $|T'_{ll}| > 0$ and $|T'_{rl}| > 0$. Since T'_l and T'_r are clustered trees, $|T'_{lr}| = |T'_{rr}| = 2^{d-2}$. Thus, exchanging T'_{lr} and T'_{rl}, we find that $|T'_r| = 2^{d-1}$. Moreover, $\Pr[T'_l \,|\, F_{\max}(T'_l)] \geq \Pr[T_l \,|\, F_{\max}(T_l)]$ by the inductive hypothesis. Since $|T_r| = |T'_r| = 2^{d-1}$, the lemma holds.
Case 4: $|T'_{ll}| = 0$ and $|T'_{rl}| > 0$. Here, T'_{llr}, T'_{lrl} and T'_{rll} are exchangeable each other, and so are T'_{lrr} and T'_{rlr}. For T'_{lrl} and T'_{rll}, four subcases arise:
Subcase 4a: $|T'_{lrl}| = |T'_{rll}| = 0$. The probabilities of the roots of T'_{lrr} and T'_{rll} are $\delta(1-\delta)^2$ and $\delta^2(1-\delta)$, respectively. Thus, since $|T'_{rll}| = 0$, even if we exchange T'_{lrr} and T'_{rll}, the probability of T' does not increase. Thus we have the lemma using the inductive hypothesis for T'_{rl}.
Subcase 4b: $|T'_{lrl}| > 0$ and $|T'_{rll}| = 0$. First, exchange T'_{lrr} and T'_{rlr}. Next, replace T'_{lr} by a clustered tree with $|T'_{lr}|$ accepting leaves. After replacing, if $|T'_{lrl}| = 0$, this case is reduced to Subcase 4a; otherwise, $|T'_{lrl}| > 0$, $|T'_{lrr}| = 2^{d-3}$, and $|T'_{rll}| = 0$. In the latter case, exchanging T'_{lrr} and T'_{rll} and further exchanging T'_{lrl} and T'_{lrr}, we see that T' is reduced to a clustered tree. Thus, it is sufficient to show that these exchanges do not increase the probability. Let $p = \Pr[T'_{lrl} \,|\, F_{\max}(T')]$ before the exchanges. Then the change of the probability by the exchanges is $-\delta(1-\delta)^2 + \delta^2(1-\delta) - \delta^2(1-\delta)p + \delta(1-\delta)^2 p = \delta(1-\delta)(1-2\delta)(p-1) < 0$, since $0 < \delta < \frac{1}{2}$ and $0 < p < 1$. This implies the lemma.
Subcase 4c: $|T'_{lrl}| = 0$ and $|T'_{rll}| > 0$. Since T'_{lr} and T'_{rl} are exchangeable, this case is reduced to Subcase 4b.

Subcase 4d: $|T'_{lrl}| > 0$ and $|T'_{rll}| > 0$. First, exchange T'_{llr} and T'_{rll}. Since $|T'_{ll}| = 0$ before the exchange, $|T'_{rll}| = 0$ after the exchange. Next, replace T'_l by a clustered tree with $|T'_l|$ accepting leaves. After replacing, if $|T'_{llr}| = 0$, this case is reduced to Subcase 4b; otherwise, $|T'_{llr}| > 0$ and $|T'_{lr}| = 2^{d-2}$. In the latter case, exchanging T'_{lrl} and T'_{rll} and further exchanging T'_{llr} and T'_{lrl}, we get a clustered tree without changing the probability. ∎

Now, we are ready to show the main theorem in this paper.

Theorem 17 *Every clustered tree is worst.*

Proof. We show the theorem by induction on the depth of the tree. Since it is clear when $d \leq 1$, we assume that $d > 1$. Let T be a given clustered tree of depth d, and T' be any worst tree of depth d with $|T'| = |T|$. Since T' is worst, it is sufficient to show that $\Pr[T' \,|\, F_{\max}(T')] \geq \Pr[T \,|\, F_{\max}(T)]$. If either T'_l or T'_r is not worst, we can improve the probability of T' by replacing them. This contradicts that T' is worst. Thus, both T'_l and T'_r are worst. By the inductive hypothesis, we can replace T'_l and T'_r by clustered trees with $|T'_l|$ and $|T'_r|$ accepting leaves, respectively. Thus, by Lemma 16, $\Pr[T' \,|\, F_{\max}(T')] \geq \Pr[T \,|\, F_{\max}(T)]$. ∎

Corollary 18 *Let T be a worst tree of depth d. Then,* $\Pr[T \,|\, F_{\max}(T)] = \sum_{i=0}^{k} \delta^i (1-\delta)^{d-t_{k-i}-i}$, *where* $|T| = \sum_{i=0}^{k} 2^{t_i}$ *with* $t_k > \cdots > t_1 > t_0 \geq 0$.

3.2 Application of a worst tree

Theorem 19 δ-PP = BPP *for* $0 < \delta < \frac{1}{2}$.

Proof. By Theorem 6, Definition 3 and 4, BPP = δ-BPP $\subseteq$ δ-PP. Thus it is sufficient to show that δ-PP $\subseteq$ BPP. Let M be a PTM with $L(M) \in \delta$-PP for some δ, and $p(n)$ be the depth of the computation path of M on input of length n. Let m be a minimum positive integer such that $(1-\delta)(1-\delta^m) \geq \frac{1}{2}$. Since $\lim_{m\to\infty}(1-\delta)(1-\delta^m) = 1-\delta > \frac{1}{2}$, m must be a constant. Without loss of generality, we assume that $p(n) > m+1$. Here we consider a worst tree T of depth $p(n)$ with $|T| \geq 2^{p(n)-1} - 2^{p(n)-m-1}$. Since $2^{p(n)-1} - 2^{p(n)-m-1} = \sum_{i=1}^{m} 2^{p(n)-i-1}$, using Corollary 18, we find that $\Pr[T \,|\, F_{\max}(T)] = (1-\delta)^2(\delta^0 + \delta^1 + \cdots + \delta^{m-1}) = (1-\delta)(1-\delta^m) \geq \frac{1}{2}$.

First, we consider an x_1 of length n with $x_1 \notin L(M)$. Since $x_1 \notin L(M)$, for its computation tree T_1 and every δ-assignment F to T_1, $\Pr[T_1 \,|\, F] < \frac{1}{2}$. Consequently, $\Pr[T_1 \,|\, F_{\max}(T_1)] < \frac{1}{2}$. Thus, Proposition 12 shows that $|T_1| < |T| = 2^{p(n)-1} - 2^{p(n)-m-1}$. Hence if M computes with a perfect random source, $\Pr[M \text{ accepts } x_1] < \frac{2^{p(n)-1} - 2^{p(n)-m-1}}{2^{p(n)}} = \frac{1}{2} - \frac{1}{2^{m+1}}$. Since m is a constant, by repeating the algorithm enough times, the probability can be improved to $\frac{1}{4}$.

Second, we consider an x_2 of length n with $x_2 \in L(M)$. For the computation tree T_2 of x_2, we denote by $\bar{T}_2$ the computation tree constructed by replacing each accepting (rejecting) leaf by a rejecting (resp. accepting) leaf. Then, $\Pr[T_2 \,|\, F] > \frac{1}{2}$ for every δ-assignment F to T_2, consequently, $\Pr[\bar{T}_2 \,|\, F_{\max}(\bar{T}_2)] < \frac{1}{2}$. Thus, using the same argument above, we have $|\bar{T}_2| < 2^{p(n)-1} - 2^{p(n)-m-1}$, or $|T_2| > 2^{p(n)} - |\bar{T}_2| = 2^{p(n)-1} + 2^{p(n)-m-1}$. Hence if M computes with a perfect random source, $\Pr[M \text{ accepts } x_2] > \frac{1}{2} + \frac{1}{2^{m+1}}$. This completes the proof. ∎

3.3 Characterizations of a worst tree

The structure of a clustered tree, as the structure of a worst tree, plays an important role in the previous subsections. Then, do clustered trees completely characterize worst trees? The answer is no.

Definition 20 *Let T be a δ-tree of depth $d \geq 0$. T is a* scattered tree *if (1) $d = 0$, or (2) $0 \leq |T_r| - |T_l| \leq 1$, and both T_r and T_l are scattered trees.*

Intuitively, a scattered tree is a tree whose all subtrees of the same depth have the same number of accepting leaves. Although the structure of a scattered tree is opposite to the structure of a clustered tree, the following corollary holds.

Corollary 21 *Every scattered tree is worst.*

Outline of proof. By Corollary 18, it is sufficient to show that for any scattered tree T of depth d, $\Pr[T\,|\,F_{\max}(T)\,] = \sum_{i=0}^{k} \delta^i (1-\delta)^{d-t_{k-i}-i}$, where $t_0, t_1, \cdots, t_k$ are the integers such that $|T\,| = \sum_{i=0}^{k} 2^{t_i}$ with $t_k > \cdots > t_1 > t_0 \geq 0$. Using an induction on the depth of the tree, we have that. ∎

Moreover, combining the definitions of clustered and scattered trees, we can define a *chimera tree* as follows:

Definition 22 *Let T be a δ-tree of depth $d \geq 0$. Then T is a* chimera tree *if (1) $d = 0$, (2) $|T_l\,| = 0$, and T_r is a chimera tree, (3) T_l is a chimera tree and $|T_r\,| = 2^{d-1}$, or (4) $0 \leq |T_r\,| - |T_l\,| \leq 1$ and both T_l and T_r are chimera trees.*

By a simple induction on the depth of the tree together with Theorem 17, Corollary 18 and 21, it is not hard to see that a chimera tree is worst. Moreover, as shown in the proof of Lemma 16, a δ-tree has many subtrees that are exchangeable each other. Exchanging these subtrees of a chimera tree, we get many variations of a worst tree. Although the author conjectures that every worst tree is a chimera tree or its variation, the complete characterization of a worst tree remains an open problem.

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