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A Prior Reduced Model of Dynamical Systems

Haoran XIE

Japan Advanced Institute of Science and Technology

Research Fellow of the Japan Society for the Promotion of Science

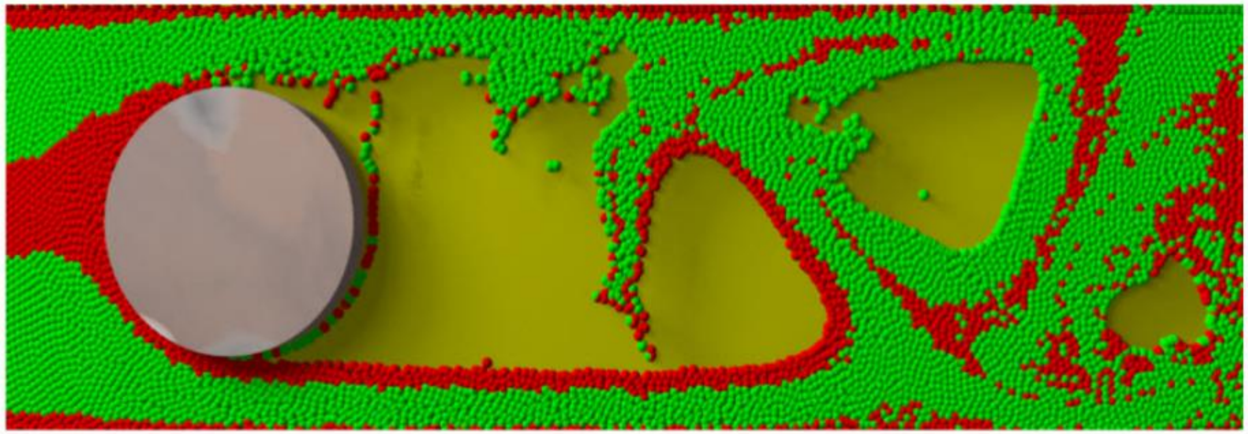
(joint work with **K.MIYATA** JAIST, **Z. WANG** and **Y. ZHAO** KSU)

PROBLEMS

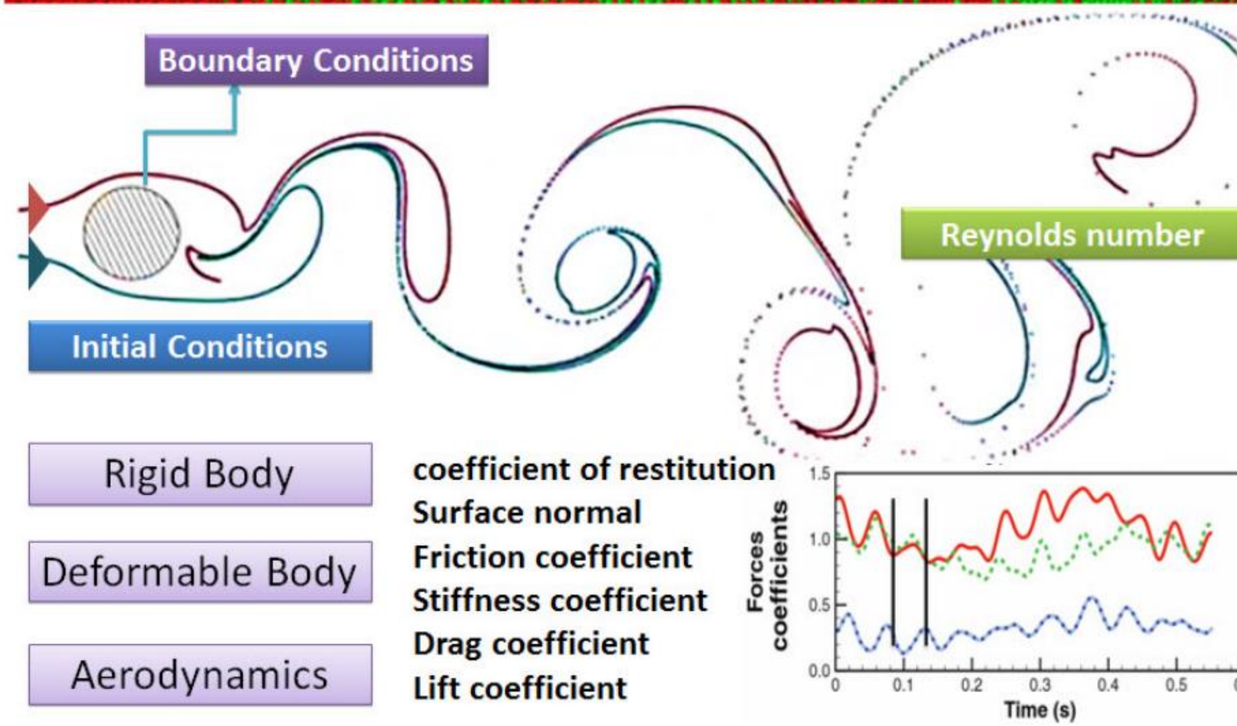
Computer Animation

- **P1** High computation cost
- **P2** Various control parameters

As shown as



※1

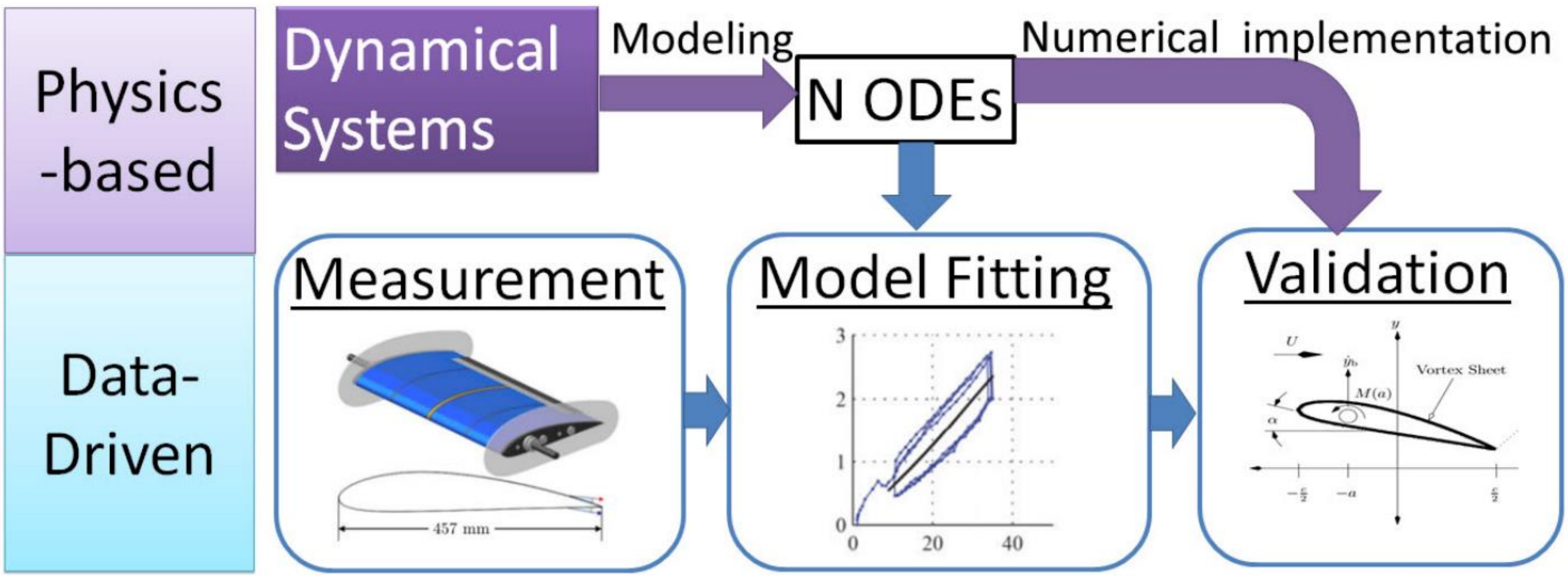


※2

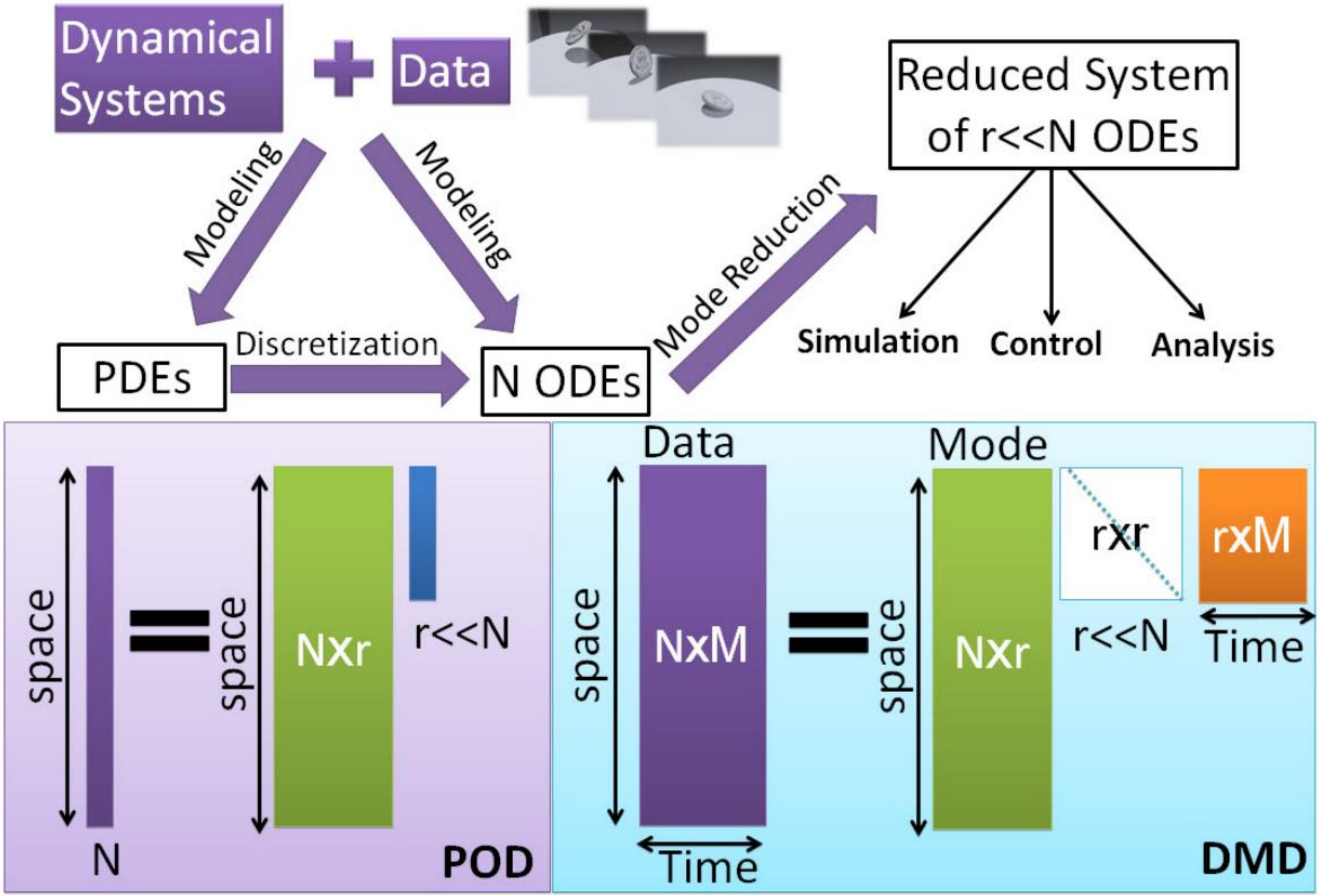
※ 1. He et.al, SIGGRAPH Asia 2012; 2. Vershnet et.al, Physical Review E. 2013

P1

TO Reduce computation cost



Data-driven to Model Reduction



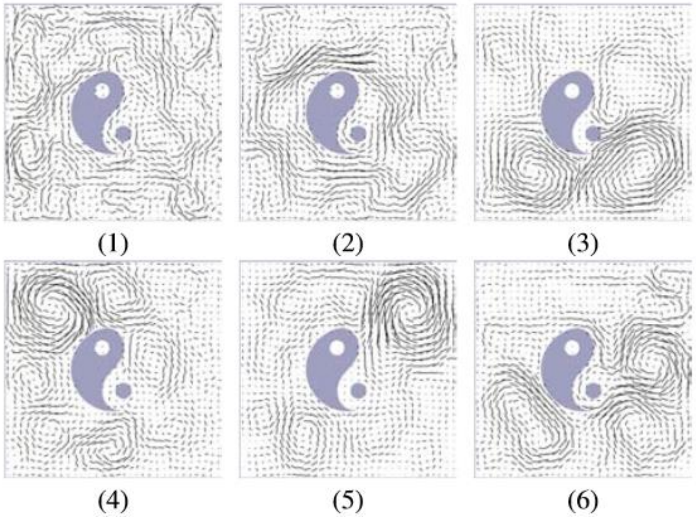
※ POD = Proper Orthogonal Decomposition; DMD = Dynamic Mode Decomposition

P. J. Schmid (2010). Dynamic mode decomposition of numerical and experimental data. Journal of Fluid Mechanics

Model Reduction in CG

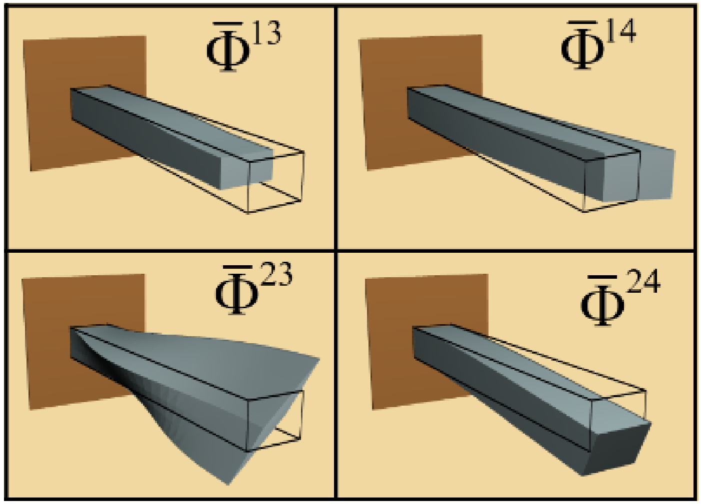
Fluid simulation

[Treuille et al., 2006]



Deformable Models

[Barbic et al., 2005]



[T. de Witt et al., 2012]

[Zhao et al., 2013]

TO find control parameters

P2

e.g., drag/lift coefficients

$$\mathbf{F}_{a\rightarrow w} = \frac{1}{2} C_D \rho_a \pi r^2 \left|\mathbf{u}_{\text{rel}}\right| \mathbf{u}_{\text{rel}}$$

$$F_{drag}^2 = \frac{C_D}{2} \rho_g \pi \tilde{R}^2 \Delta u^2$$

$$\mathbf{f}_i^{drag} = -k_{drag} r_i^2 \left|\mathbf{v}_i - \mathbf{u}_i\right| (\mathbf{v}_i - \mathbf{u}_i)$$

$$\mathbf{f}_i^{lift} = -k_{lift} V_i (\mathbf{v}_i - \mathbf{u}_i) \times \omega_i,$$

$$F_D = \frac{C_D}{2} \rho_L \pi r^2 \mathbf{w}_{\text{rel}} \left|\mathbf{w}_{\text{rel}}\right|$$

$$\mathbf{f}_i^{\text{drag}} = -k_{\text{drag}} r_i^2 \left|\mathbf{v}_{\oplus} - \mathbf{u}_{\oplus}\right| (\mathbf{v}_{\oplus} - \mathbf{u}_{\oplus}).$$

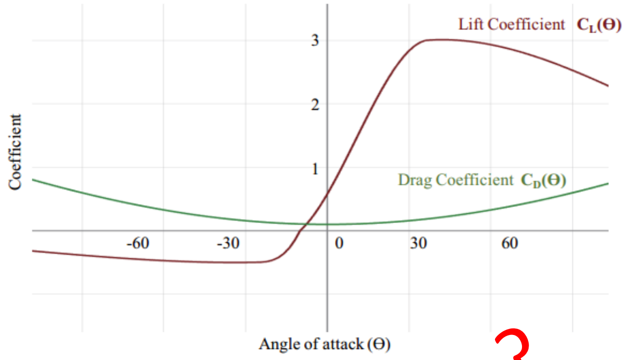
[SCA06,S09,S13,EG09,TVC12]

Constant

Reynolds number

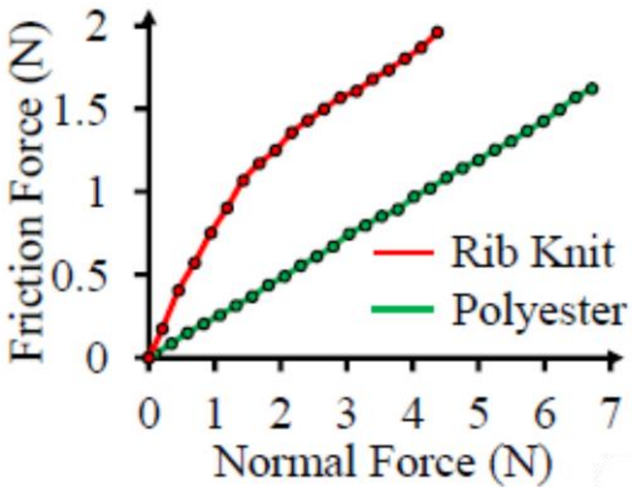
Angle of attack

$$C_D = \frac{24}{Re^{0.72}}$$

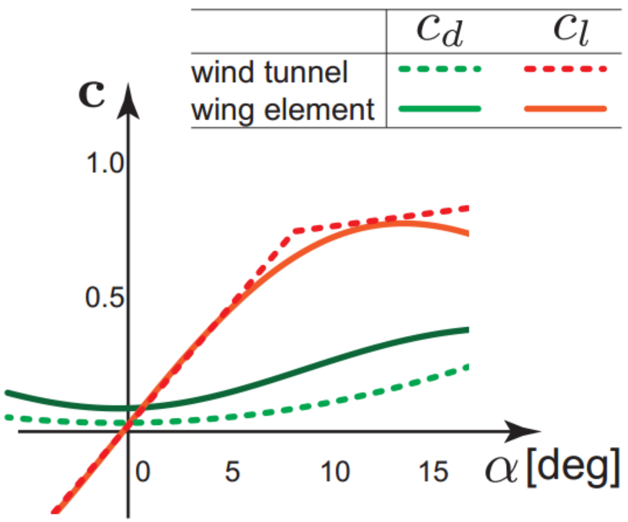


Which is proper data?

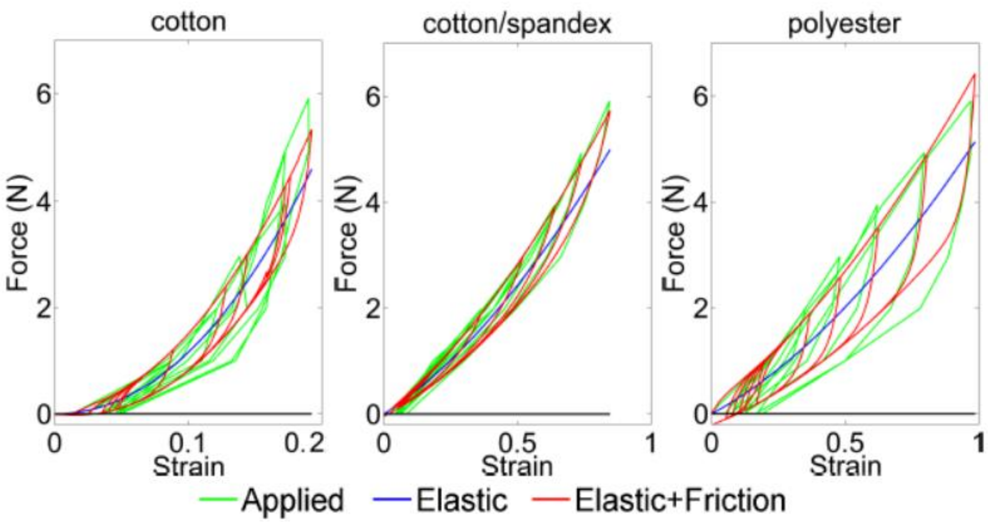
Data-driven to Parameter Estimation



Modeling Friction and Air Effects between Cloth and Deformable Bodies,S2013



Interactive Design and Optimization of Free-formed Free-flight Model Airplanes,S2014



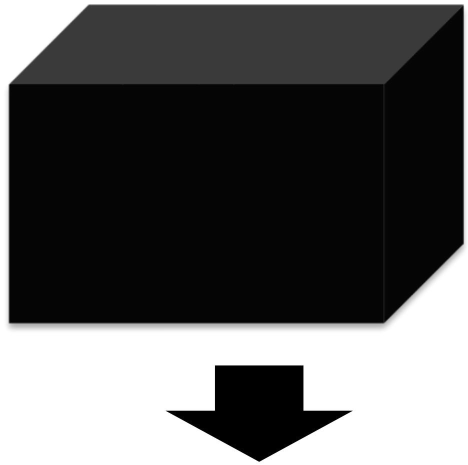
Modeling and Estimation of Internal Friction in Cloth, SA2013

Why others is improper?

Curse of Dimensionality

Dynamical Systems

INPUTS:
Parameter1 N nodes
Parameter2 N nodes
Parameter3 N nodes
···



$N \times N \times N \cdots$

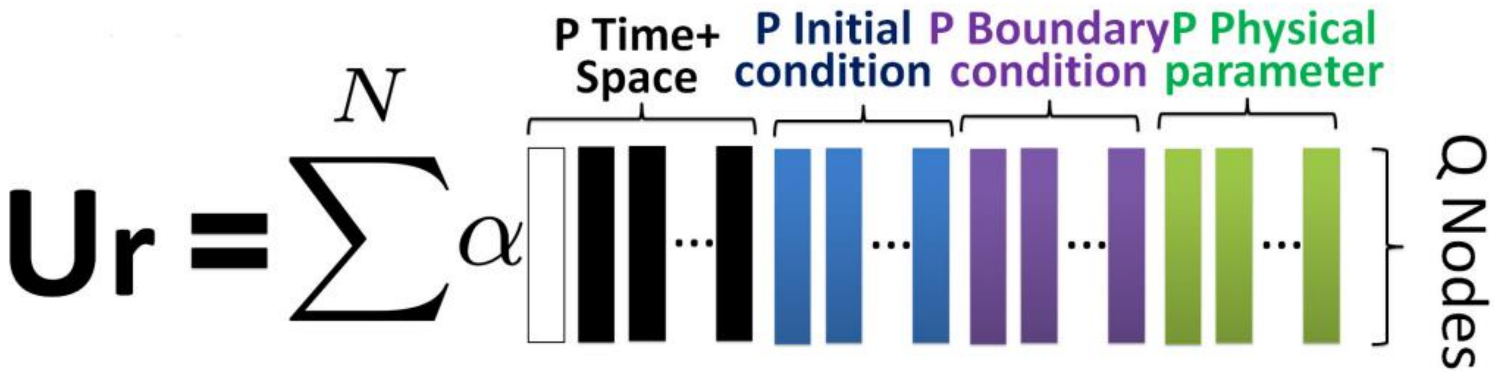
A Prior Reduced Model of Dynamical Systems

SEPARATED REPRESENTATION

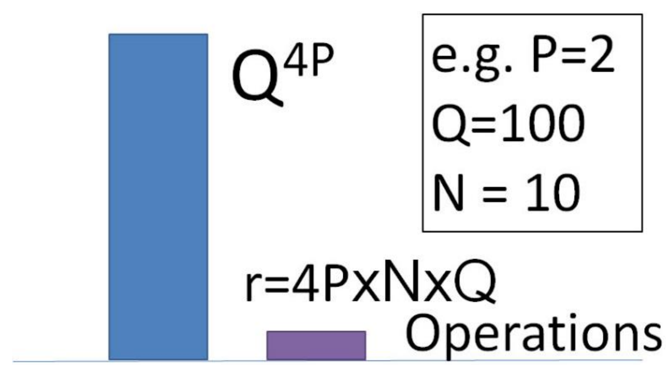
Separated Representation

Dynamical system: $D(U) = G(U, P)$

State: $U(x_1, x_2, ..., x_d) = \sum_{i=1}^{N} \alpha_i \prod_{j=1}^{d} U_i^j(x_j)$



- Model-based (no precomputed snapshots)
- Reduced Model



Reduction Solver

- Determine prior unknown functions $U_i^j(x_j)$

step n.

known

Test functions: $U = \sum_{i=1}^{n-1} \alpha_i \prod_{j=1}^{d} U_i^j(x_j) + \prod_{j=1}^{d} U_n^j(x_j)$

$\alpha_n = 1$

Weak form: $\langle D(U), U_n^j \rangle_{\Omega_j} = \langle G, U_n^j \rangle_{\Omega_j}$

Enrichment step:
Alternating directions fixed-point method

Iteration

p-th iteration

$(u_{p-1}^2, u_{p-1}^3, \ldots, u_{p-1}^d) \Rightarrow u_p^1$

...

$(u_p^1, \ldots, u_p^{k-1}, u_{p-1}^{k+1}, \ldots, u_{p-1}^d) \Rightarrow u_p^k$

Check convergence

Reduction Solver

Determine α_n

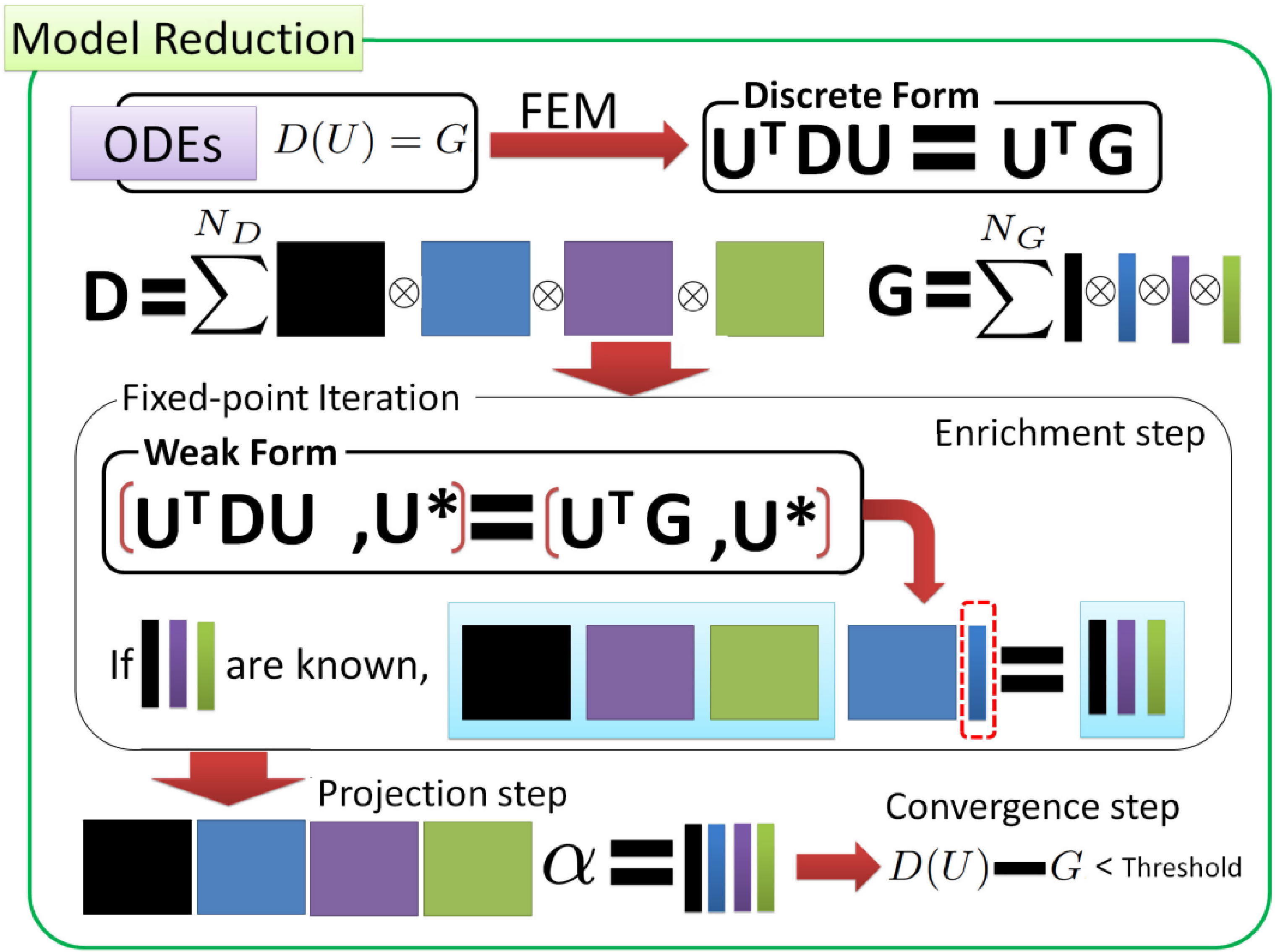
Projection step: $\langle D(U), \prod_{j=1}^{d} U_i^j(x_j)\rangle = \langle G, \prod_{j=1}^{d} U_i^j(x_j)\rangle$

Check convergence:

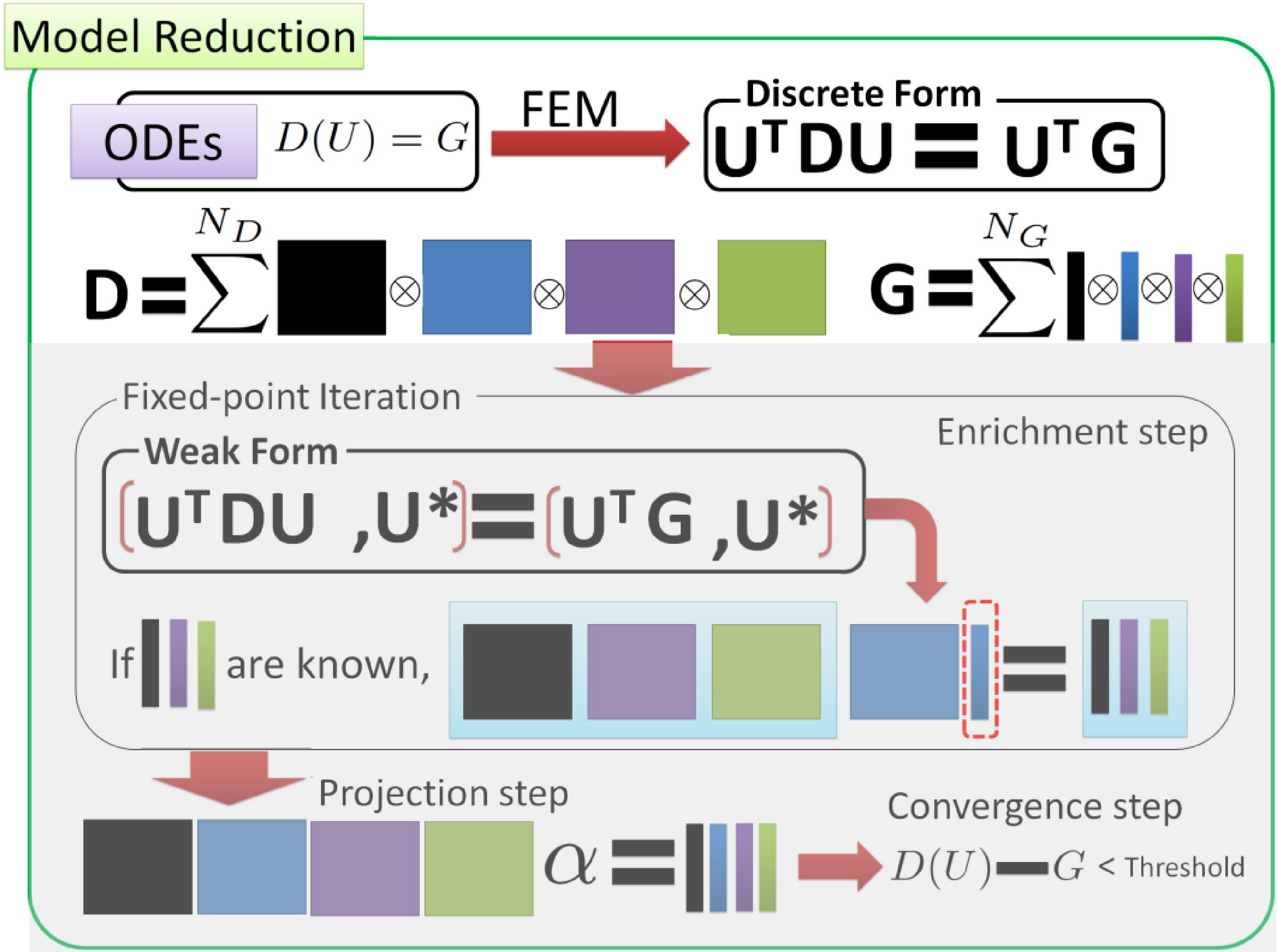
Residual term

$$R^n = D(\sum_{i=1}^{n-1} \alpha_i \prod_{j=1}^{d} U_i^j(x_j) + \prod_{j=1}^{d} U_n^j(x_j)) - G \quad < \quad \epsilon$$

A practical framework



Initialization



Discrete Formulation

An example

$$\frac{du}{dt} + ku = 0$$

$$u(t,k) = \sum_{i=1}^{N} T_i(t) K_i(k)$$

in $\Omega_t \times \Omega_k$

Weak form: $\langle D(U), U_n^j \rangle_{\Omega_j} = \langle G, U_n^j \rangle_{\Omega_j}$

$$\langle \frac{dT_n}{dt}, T_n \rangle \langle K_n, K_n \rangle + \langle T_n, T_n \rangle \langle kK_n, K_n \rangle = -\sum_{i=1}^{n-1} (\langle \frac{dT_i}{dt}, T_n \rangle \langle K_i, K_n \rangle + \langle T_i(t), T_n \rangle \langle kK_i(k), K_n \rangle)$$

$$T^T PT \cdot K^T M_k K + T^T M_t T \cdot K^T NK = -\sum_{i=1}^{n-1} (T_i^T PT \cdot K_i^T M_k K + T_i M_t T \cdot K_i^T NK)$$

$$\begin{cases} P_{ij} = \int_{\Omega_t} \frac{dN_i}{dt} N_j dt \\ N_{ij} = \int_{\Omega_k} N_i k N_j dk \\ M_t = \int_{\Omega_t} N_i N_j dt \\ M_k = \int_{\Omega_k} N_i N_j dk \end{cases}$$

shape functions

Algebraic Formulation

$$T^T PT \cdot K^T M_k K + T^T M_t T \cdot K^T NK = -\sum_{i=1}^{n-1} (T_i^T PT \cdot K_i^T M_k K + T_i M_t T \cdot K_i^T NK)$$

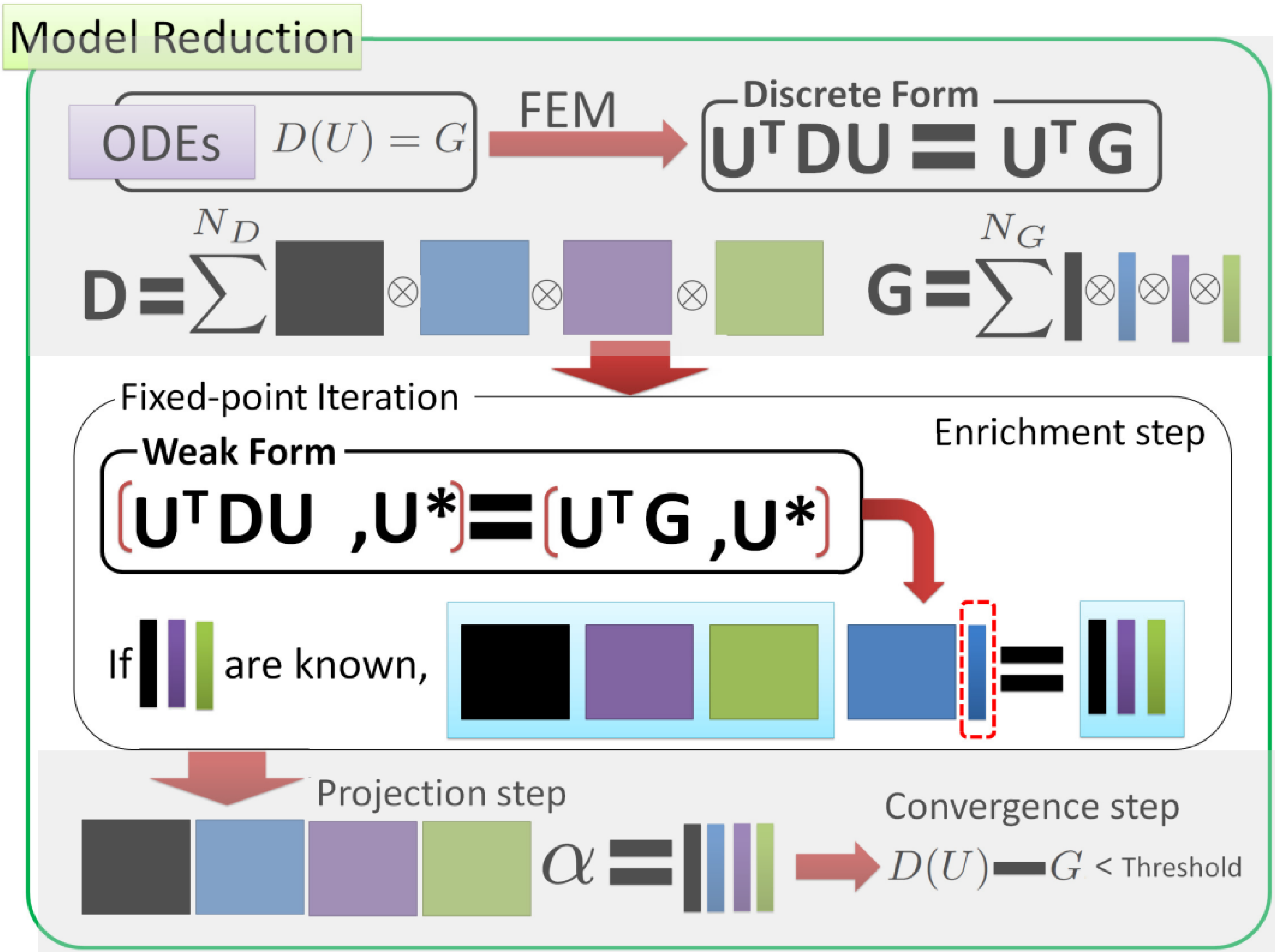
$$D(U) = G(U, P) \Rightarrow \mathbf{U^T D U = U^T G}$$

$$U = \sum_{i=1}^{N} \alpha_i U_1^i \otimes U_2^i \ldots \otimes U_d^i \qquad \boxed{N_j}$$

$$D = \sum_{i=1}^{N_D} D_1^i \otimes D_2^i \ldots \otimes D_d^i \qquad \boxed{N_j \times N_j}$$

$$G = \sum_{i=1}^{N_G} G_1^i \otimes G_2^i \ldots \otimes G_d^i \qquad \boxed{N_j}$$

Enrichment Step



Enrichment step

Test functions: $U_n = \alpha_n R_1 \otimes R_2 ... \otimes R_d$

$\alpha_n = 1$

$U^T D U = U^T G$

$$\sum_{i=1}^{N_D} D_1^i R_1 \otimes D_2^i R_2 ... \otimes D_d^i R_d = G - \sum_{i=1}^{N_D} \sum_{k=1}^{n-1} \alpha_i D_1^i U_1^k \otimes D_2^i U_2^k ... \otimes D_d^i U_d^k$$

- Fixed-point method

p-th iteration $(R_1^p, .., R_{j-1}^p R_{j+1}^{p-1}, ... R_d^{p-1}) \Rightarrow R_j^p$

$$ER_j = \sum_{i=1}^{N_G} \left(\prod_{k=1, k \neq j}^{d} R_k^T G_k^i \right) G_j^i - \sum_{i=1}^{N_D} \sum_{k=1}^{n-1} \left(\prod_{m=1, m \neq j}^{d} R_m^T D_m^i U_m^i \right) \alpha^k D_j^i U_j^k$$ ⇐ Weak form

where $E = \sum_{i=1}^{N_D} \left(\prod_{k=1, k \neq j}^{d} R_k^T D_k^i R_k \right) D_j^i$

Enrichment step

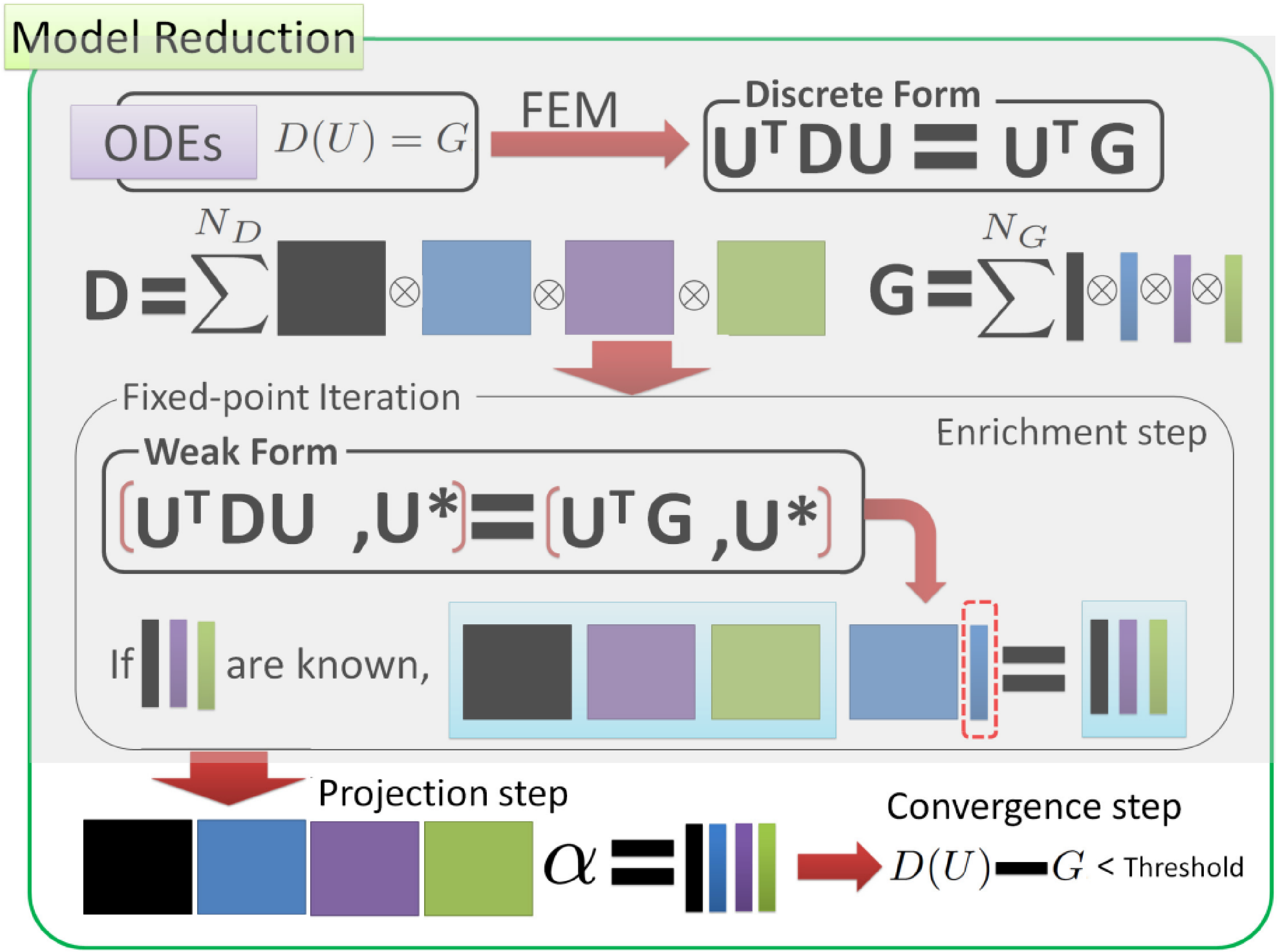
Check convergence:

$$\|R_1^p \otimes R_2^p \ldots \otimes R_d^p - R_1^{p-1} \otimes R_2^{p-1} \ldots \otimes R_d^{p-1}\| < \epsilon$$

$$\Downarrow$$

$$U = \sum_{i=1}^{N} \alpha_i U_1^i \otimes U_2^i \ldots \otimes U_d^i \qquad U_j^n = \frac{R_j}{\|R_j\|}$$

Projection Step



Projection Step

Determine α_n

$$\langle D(U), \prod_{j=1}^{d} U_i^j(x_j)\rangle = \langle G, \prod_{j=1}^{d} U_i^j(x_j)\rangle$$

$$\boxed{BA = H}$$

$$A = [\alpha_1 \alpha_2 ... \alpha_n]^T$$

$$B_{ij} = \sum_{k=1}^{N_D} [\prod_{e=1}^{d} (U_e^i)^T D_e^k U_e^j]$$

$$H_i = \sum_{m=1}^{N_G} [\prod_{k=1}^{d} (U_k^i)^T G_k^m]$$

$$1 \leq i, j \leq n$$

Check convergence:

$$R^n = \sum_{i=1}^{N_D} \sum_{k=1}^{n} \alpha_i D_1^i U_1^k \otimes D_2^i U_2^k ... \otimes D_d^i U_d^k - G \ < \ \epsilon$$

Coupled Model

$$U = \sum_{i=1}^{N} \alpha_i U_1^i \otimes U_2^i \ldots \otimes U_d^i + R_1 \otimes R_2 \ldots \otimes R_d$$

$$W = \sum_{i=1}^{N} \beta_i W_1^i \otimes W_2^i \ldots \otimes W_d^i + S_1 \otimes S_2 \ldots \otimes S_d$$

$$\begin{cases} D_U(U,W) = G_U \\ D_W(U,W) = G_W \end{cases} \xrightarrow{\text{decoupling}} \begin{cases} D_U(U,W,R,S=0) = G_U \\ D_W(U,W,R=0,S) = G_W \end{cases}$$

A practical framework

Model Reduction

Initialize D, G, and U
for $E = 1$ to E_{max} **do** // number of coupled equations
  for $N = 1$ to N_{max} **do** // number of enrichments
    for $p = 1$ to p_{max} **do** // fixed-point iteration
      Compute R_j^p // enrichment step
      Check convergence
    end for
    Normalize U_j^n
    Compute coefficients α_i // projection step
    Update U_n
    Check convergence
  end for
  Update G
end for

$\alpha =$ → $D(U) - G$ < Threshold

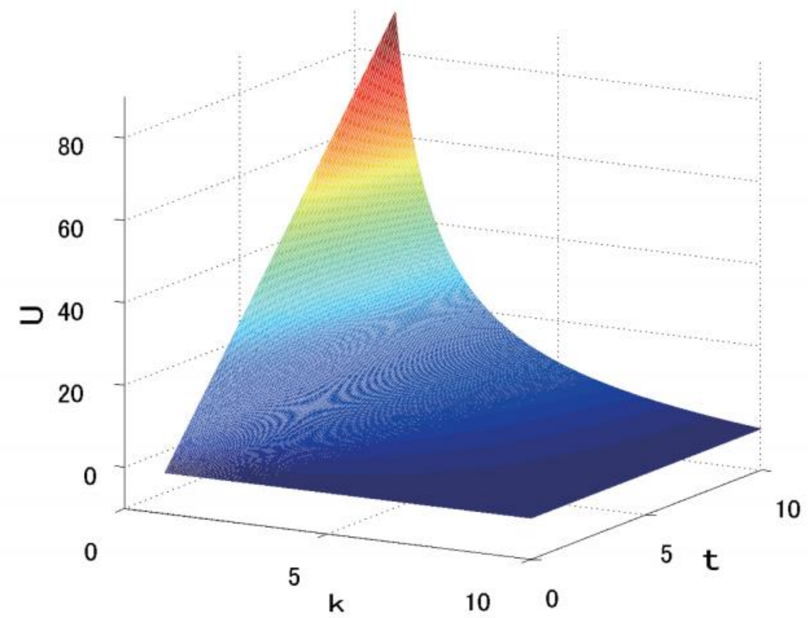
RESULTS

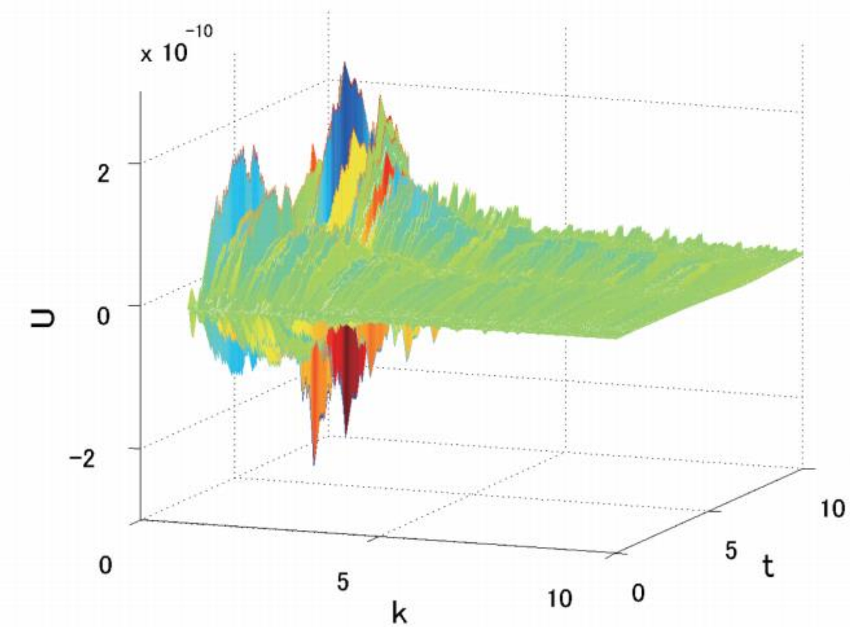
Parametric model :Linear

$$k(\frac{du}{dt}+1)=10$$

$$u(t=0)=0$$

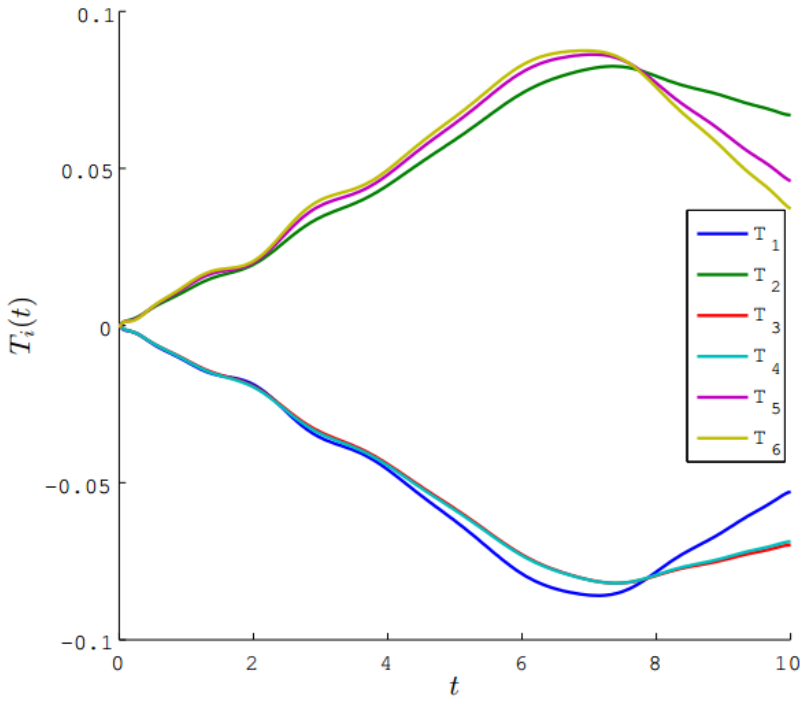
$$u(t,k)=\sum_{i=1}^{n}\alpha_i T_i(t)K_i(k)$$

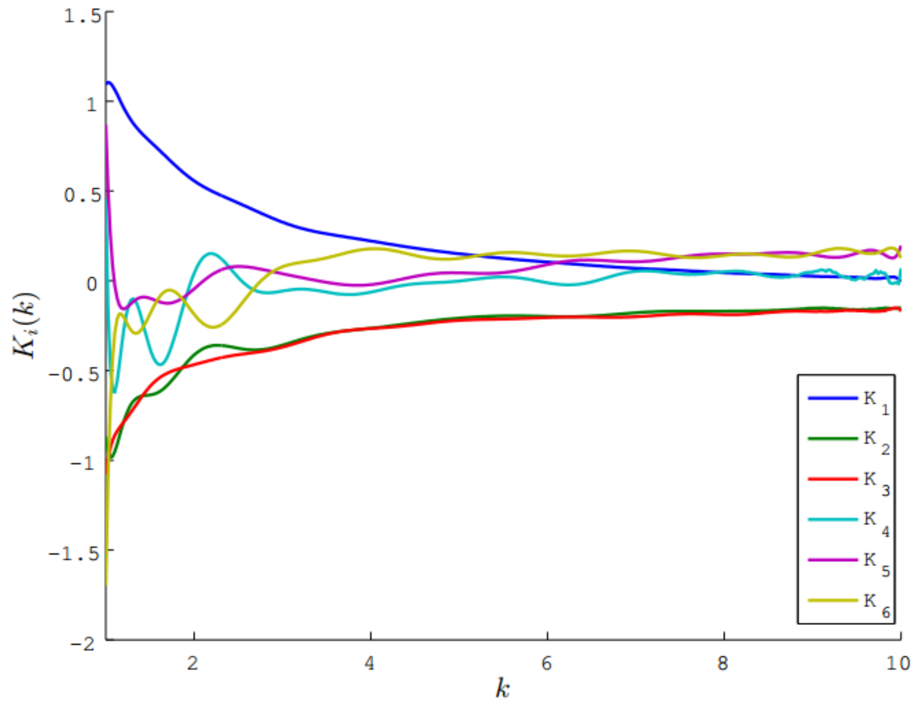




Parametric model: Linear

- Base functions



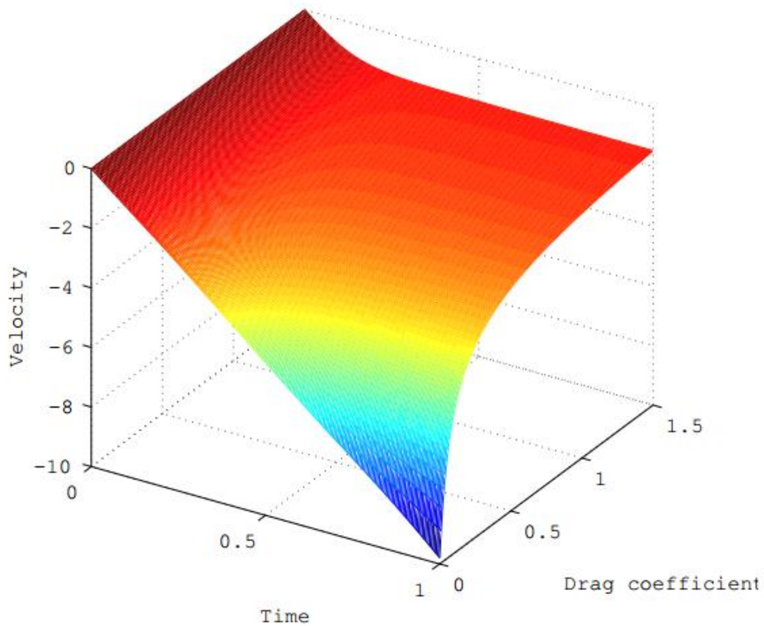


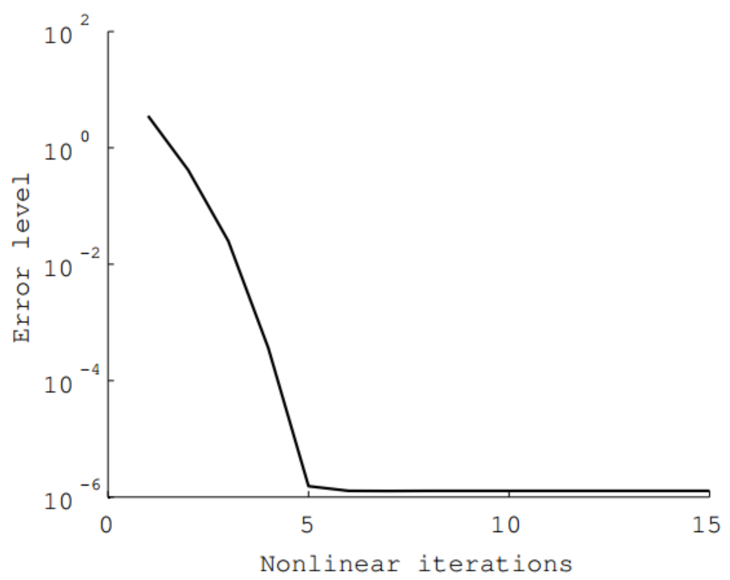
Parametric model :Nonlinear

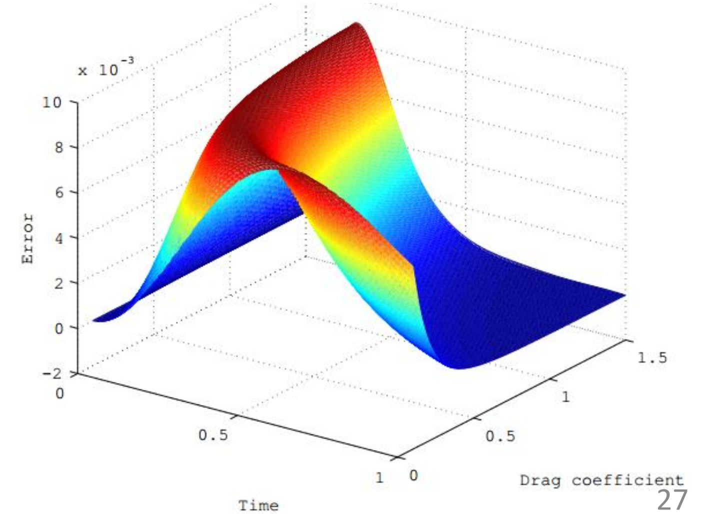
1D free-fall problem

$$\frac{du}{dt} = \frac{1}{2m}\rho_f C_D A u^2 - g$$

$$u(t, C_D) = \sum_{j=1}^{n} \alpha_j T_j(t) D_j(C_D)$$





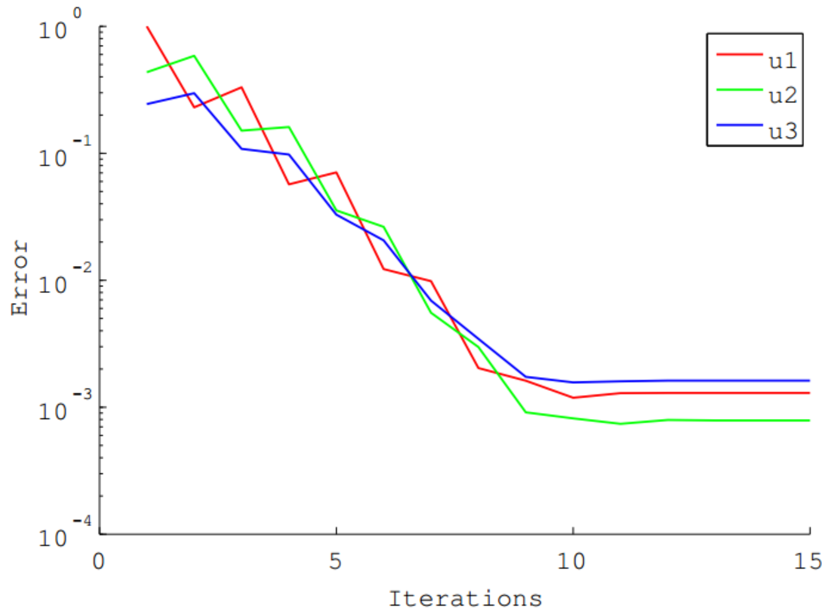


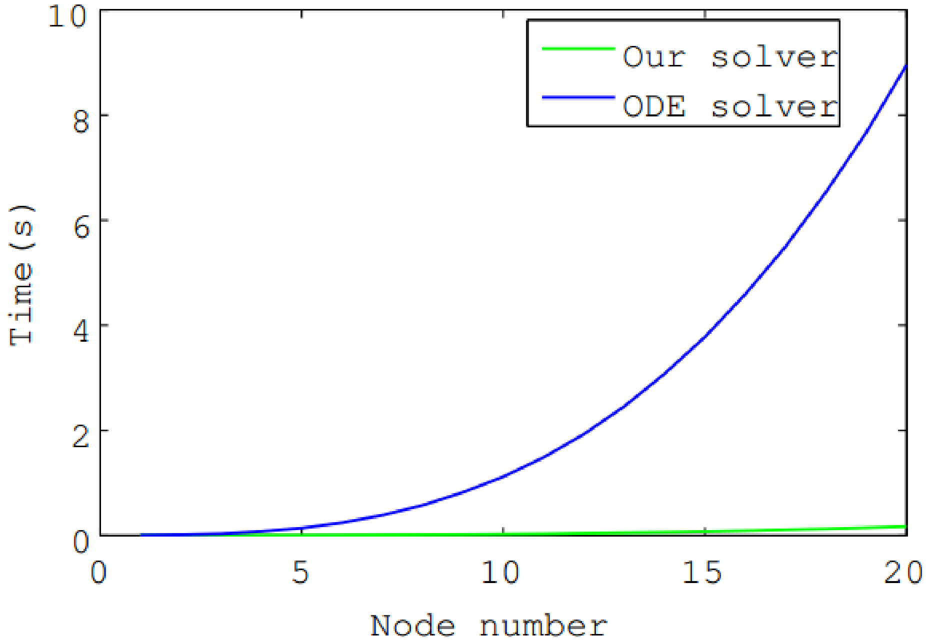
Coupled Model

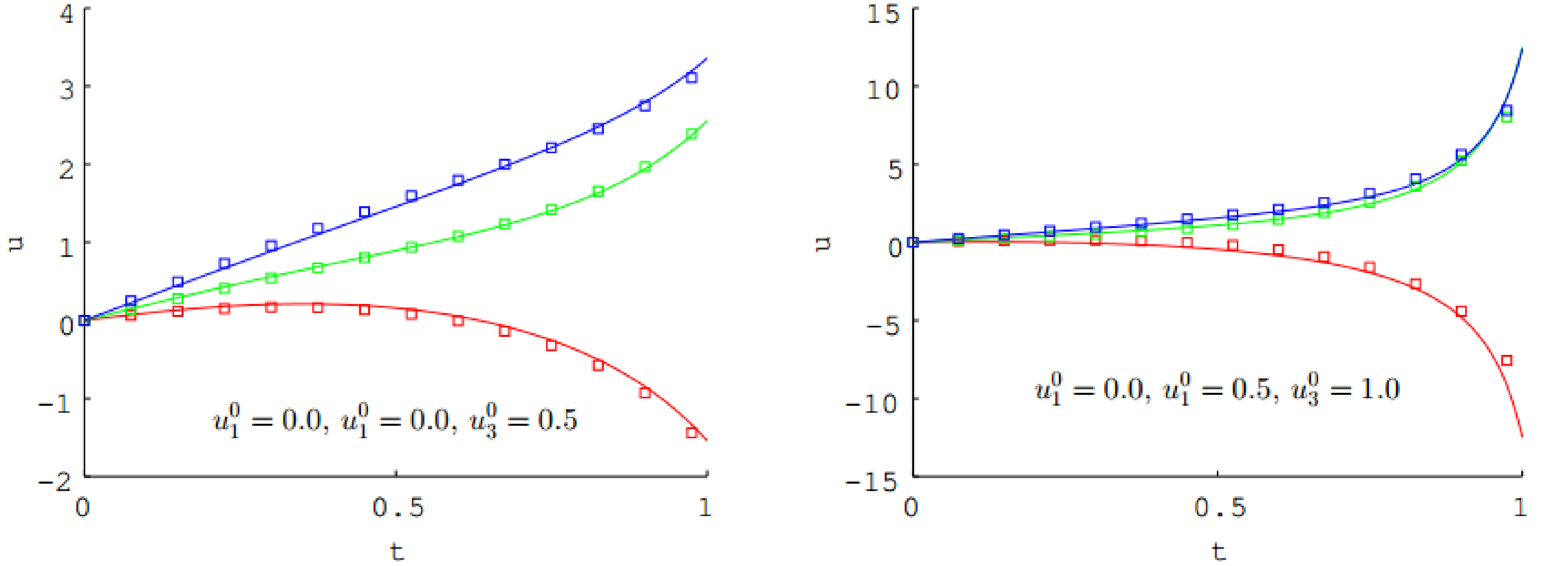
$$\begin{cases} \frac{du_1}{dt} + u_2 u_3 = 1 \\ \frac{du_2}{dt} + u_1 u_3 = 2 \\ \frac{du_3}{dt} + u_1 u_2 = 3 \end{cases}$$

$$u_i(t=0) = u_i^0, i = 1, 2, 3$$

$$\boxed{\hat{u}_i(t, u_i^0) = \sum_{j=1}^{n} \alpha_j T_j(t) U_j^1(u_1^0) U_j^2(u_2^0) U_j^3(u_3^0)}$$

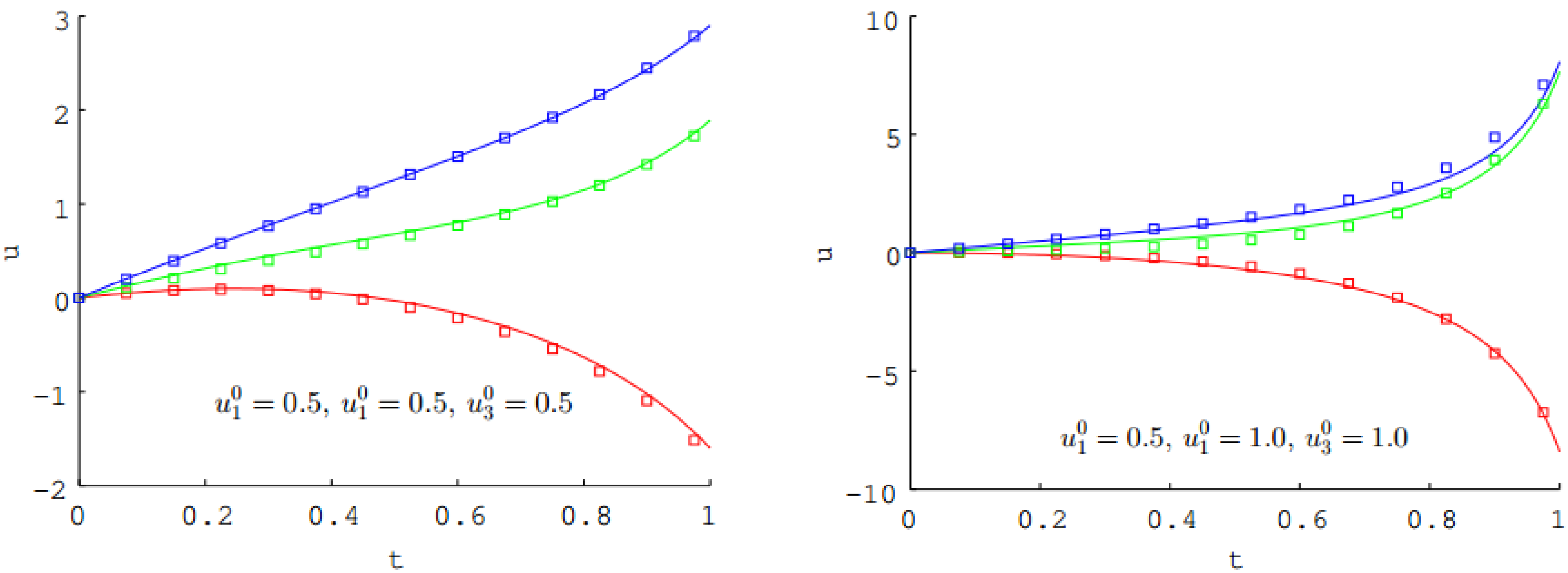






u
t
$u_1^0 = 0.0,\ u_1^0 = 0.0,\ u_3^0 = 0.5$
$u_1^0 = 0.0,\ u_1^0 = 0.5,\ u_3^0 = 1.0$

(a) (b)



u
t
$u_1^0 = 0.5,\ u_1^0 = 0.5,\ u_3^0 = 0.5$
$u_1^0 = 0.5,\ u_1^0 = 1.0,\ u_3^0 = 1.0$

Coupled Model: Stiff

Prey–Predator model

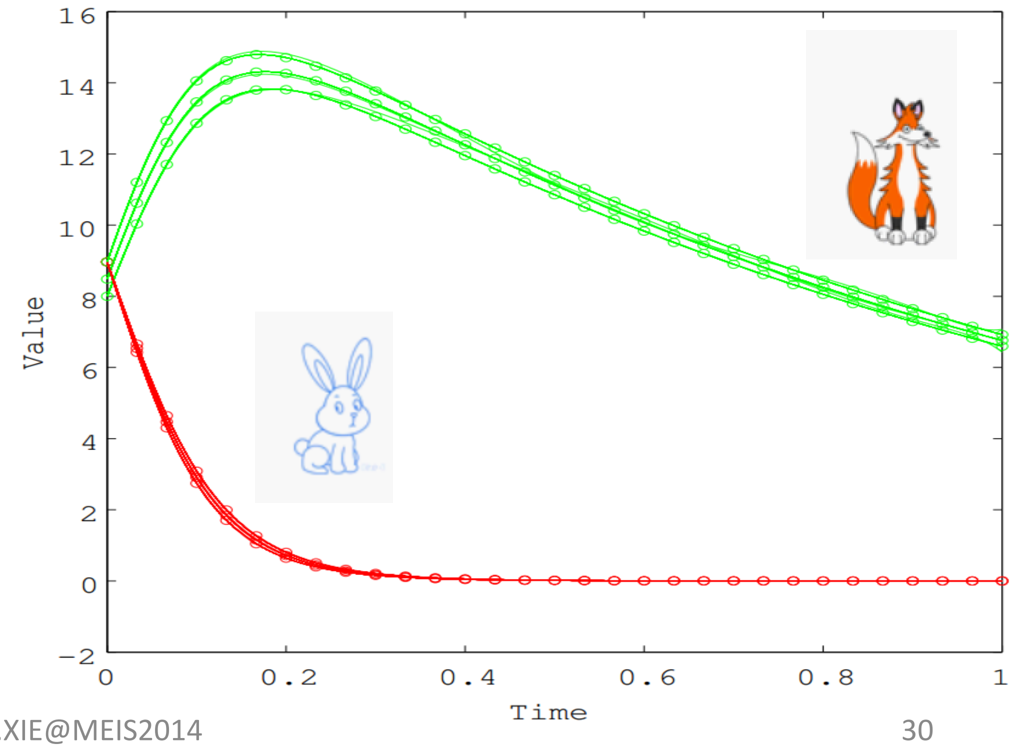
Lotka–Volterra equations:

$$x(t, x_0, y_0) = \sum_{i=1}^{n} \alpha_i T_i(t) X_i(x_0) Y_i(y_0)$$

$$y(t, x_0, y_0) = \sum_{j=1}^{n} \alpha_j T_j(t) X_j(x_0) Y_j(y_0)$$

$$x' = x(a - by)$$

$$y' = -y(c - dx)$$

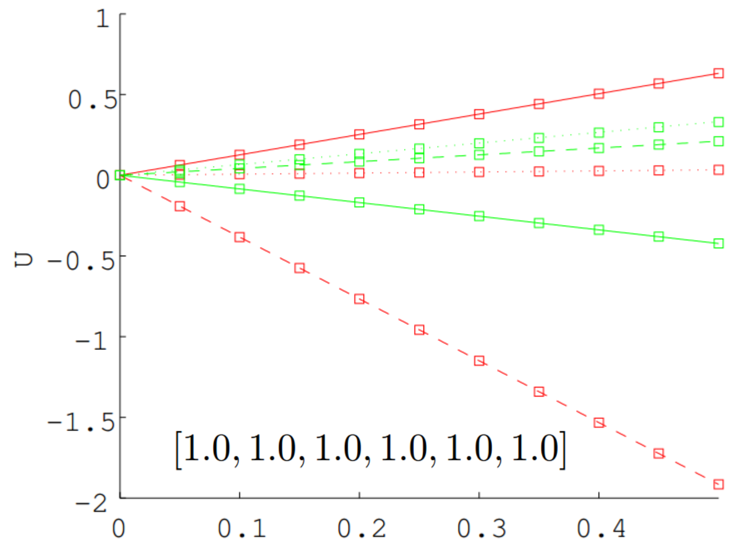


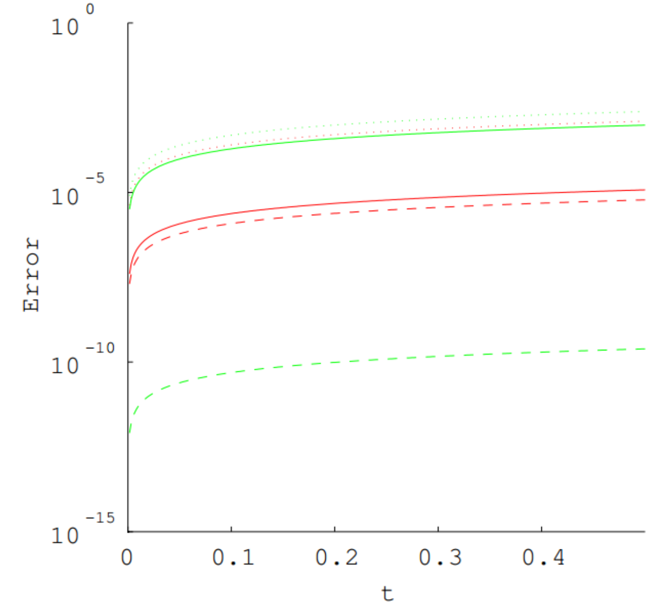
Coupled Model: Complex

Underwater rigid body dynamics
(without velocity coupling)

$$\begin{cases}(mE+M)\frac{du}{dt}=(mE+M)u\times\omega+f_g\\(J+I)\frac{d\omega}{dt}=(J+I)\omega\times\omega+(Mu)\times u+\tau_g\end{cases}$$

$$U_k(t,u_k^0)=\sum_{i=1}^{n}\alpha_i T_i(t)\prod_{j=1}^{6}U_i^j(u_j^0)$$

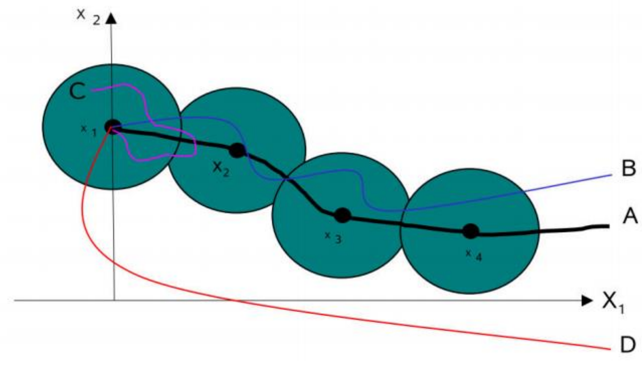




Red: Translational velocity
Green: Angular velocity

Conclusion

- A practical framework of separated representation
- A promising model-driven approach for computer animation
- Reduced Model for weakly-coupled and nonlinear models



Limitation and Future work

◆ Graphics application in progress

◆ Strongly coupled and nonlinear model

  ◆ Nonlinear Model Reduction (Trajectory PieceWise-Linear approach (TPWL), Discrete Empirical Interpolation Method)

  ◆ Hybrid reduced approach with **POD**

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H.XIE@JAIST

THANK YOU